# The Binomial Theorem

The Binomial Theorem is a powerful algebraic framework used to expand polynomial expressions of the form $(a + b)^n$, where n is a positive integer. Instead of performing tedious, repetitive polynomial long multiplication, this theorem allows engineers to directly compute the complete expansion. The resulting expansion consists of a series of terms involving $a^x b^y$, where the sum of the exponents x and y for any individual term is always strictly equal to the total power n.

### The General Expansion Formula

For any positive integer n, the algebraic expansion of a binomial expression follows a predictable structural pattern. The general mathematical expression is defined as:

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ .\ .\ .\ + b^n$$

When analyzing this mathematical series, several critical structural observations must be kept in mind for board examinations:

- **Total Number of Terms:** The total number of terms in the completely expanded polynomial is always exactly equal to n + 1.
- **Boundary Conditions:** The first term of the expansion is always $a^n$, and the final term of the expansion is always $b^n$.
- **Behavior of Exponents:** Moving from left to right across the expansion, the exponent of the first variable (a) decreases by exactly 1 for each successive term, while the exponent of the second variable (b) increases by exactly 1.
- **Sum of Exponents:** For every single individual term within the expansion, the sum of the exponent of a and the exponent of b remains constant and is always equal to n.
- **Symmetry of Coefficients:** The numerical coefficients of the terms are perfectly symmetrical from the left side of the expansion to the right side.

# Pascal's Triangle

| | |
|---|---|
| 1<br>1 1<br>1 2 1<br>1 3 3 1<br>1 4 6 4 1<br>1 5 10 10 5 1<br>1 6 15 20 15 6 1 | Pascal's Triangle provides a simple geometric arrangement of integers that yields the numerical coefficients for a binomial expansion without requiring explicit combinatorial calculations. Starting with a 1 at the top apex, each subsequent row is generated by placing a 1 at the outer boundaries and finding the interior numbers by adding the two adjacent numbers directly above them. |

For instance, looking at the bottom row shown above, the coefficients correspond precisely to a binomial raised to the seventh power (n = 7).

**Illustration:** Consider the expansion of $(a - b)^7$.

Because the second term is negative, the signs of the expanded terms must alternate, starting with a positive value for the first term.

$$(a - b)^7 = a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$$

**Finding a Particular Term in a Binomial Expansion**

In many engineering exam problems, you are not required to expand the entire polynomial but rather to find one specific term, such as the $r^{th}$ term. To locate a specific term directly, the mathematical relation isolates the combinatorial coefficient alongside the specific product of the variable powers. The general formula to find the $r^{th}$ term is modeled as:

$$r^{th} term = \frac{n(n-1)(n-2)...(n-r+2)}{(r-1)!} a^{n-r+1} b^{r-1}$$

This relationship can be simplified into standard combinatorial notation:

$$r^{th} term = {}^{n}C_{r-1}\, a^{n-r+1} b^{r-1}$$

Within this calculation, the exponent of the second variable (b) is always exactly 1 lesser than the numerical position number (r - 1).

**Illustration:** Find the $5^{th}$ term of the expansion $(a - b)^7$.

Here, let n = 7, r = 5, the first term be $a$, and the second term be $(-b)$. Applying the combinatorial tracking formula:

$$5^{th} term = {}^{7}C_{5-1}\, a^{7-5+1}(-b)^{5-1} = 35a^3b^4$$

**Sum of Coefficients**

A frequent shortcut problem on board examinations asks for the sum of all numerical coefficients in a expanded polynomial expression. Instead of fully expanding the expression and adding the numbers manually, a mathematical shortcut can be applied: substitute a value of 1 for all variables present in the binomial expression.

**Illustration:** Find the sum of the coefficients of the expansion $(x - 2y)^7$.

By substituting x = 1 and y = 1 directly into the unexpanded binomial expression, the calculation is simplified to:

$$\text{Sum of Coefficients} = [1 - 2(1)]^7 = -1$$

Using this method avoids lengthy manual expansion and provides the exact sum of the coefficients instantly.

## Quadratic Equations and Polynomial Theorems

A comprehensive mastery of polynomial functions is essential for engineering board examinations. This section focuses on second-order polynomial equations, the standard analytical properties governing their solutions, and the fundamental algebraic theorems used to evaluate remainders during polynomial division.

### Quadratic Equations of the Second Order

A quadratic equation is a second-order polynomial equation containing a single variable raised to a maximum power of two. The standard mathematical arrangement of a quadratic equation is defined as: $ax^2 + bx + c = 0$

In this standard expression, $x$ represents the unknown variable, while $a$, $b$, and $c$ represent constant coefficients, with the mathematical restriction that the leading coefficient $a \neq 0$. The solutions to this equation, commonly referred to as the roots ($x_1$ and $x_2$), represent the values of $x$ that satisfy the equality. These roots can be calculated directly by substituting the coefficients into the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

### Vieta's Formulas: Relationships Between Roots and Coefficients

Board exam problems frequently require evaluating expressions involving the roots without explicitly solving for $x_1$ and $x_2$ individually. This is achieved using relationships derived directly from the expansion of $(x - x_1)(x - x_2) = 0$:

- **Sum of Roots:** The algebraic addition of the two roots is always equal to the negative ratio of the linear coefficient to the leading coefficient: $x_1 + x_2 = \frac{-b}{a}$
- **Product of Roots:** The algebraic multiplication of the two roots is always equal to the direct ratio of the constant term to the leading coefficient: $x_1 \cdot x_2 = \frac{c}{a}$

### The Discriminant and the Nature of Roots

The term isolated underneath the radical sign in the quadratic formula determines the algebraic characteristics, or "nature," of the roots. This parameter is known as the **discriminant** (D):

$$D = b^2 - 4ac$$

By evaluating the numerical value of D, engineers can instantly classify the behavior of the polynomial's solutions:

| Discriminant Condition | Nature of the Roots |
|---|---|
| $D = 0$ | Equal, Real, and Rational (One unique real repeating root) |
| $D > 0$ and is a perfect square | Unequal, Real, and Rational |
| $D > 0$ and not a perfect square | Unequal, Real, and Irrational |
| $D < 0$ | Unequal and Imaginary (Complex conjugate pairs) |

**Mathematical Review of Number Systems:**

- **Rational Numbers:** Numbers that can be explicitly written as the quotient or fraction $\frac{p}{q}$ of two integers, where the denominator is non-zero. Examples include precise fractions like $\frac{2}{3}$, negative integers like -5, terminating decimals, or repeating decimals such as 0.1212. . . and 0.4444. . ..
- **Irrational Numbers:** Real numbers that cannot be represented as a simple fraction of two integers. Their decimal expansions are non-terminating and non-repeating. Standard examples include $\sqrt{2}$, π, and Euler's number $e$.

### The Remainder Theorem

The Remainder Theorem provides a mathematical shortcut for determining the remainder when a polynomial expression is divided by a first-degree linear binomial, avoiding the need for polynomial long division or synthetic division.

### Mathematical Derivation

When a polynomial function f(x) is divided by a linear divisor of the form (x - r), the algebraic operation can be modeled using a quotient polynomial Q(x) and a constant remainder R:

$$\frac{f(x)}{x-r} = Q(x) + \frac{R}{x-r}$$

Multiplying the entire equation by the linear denominator (x - r) yields the fundamental polynomial division identity:

$$f(x) = Q(x)(x - r) + R$$

To isolate the constant remainder R, we substitute x = r into the identity. This substitution eliminates the quotient term because (r - r) = 0

$$f(r) = Q(r)(0) + R$$

$$f(r) = R$$

Therefore, the remainder R resulting from dividing a polynomial f(x) by (x - r) is exactly equal to the polynomial evaluated at r (f(r)).

**Illustration:** Find the remainder when the polynomial (x) = $x^4 - 5x^3 + 7x^2 - 3x + 6$ is divided by the linear expression x + 1.

1. Identify the value of r by setting the linear divisor equal to zero: x + 1 = 0 ⇒ r = -1.
2. Substitute r = -1 into the polynomial function to compute f(-1):

$$R = f(-1) = (-1)^4 - 5(-1)^3 + 7(-1)^2 - 3(-1) + 6$$

3. The remainder of the polynomial division is **R = 22**

## Progressions and Series

A progression is a sequenced mathematical series of numbers that follows an explicit, predictable rule for computing successive terms. In engineering board examinations, problems involving progressions test your ability to determine individual terms, accumulate sums, and evaluate mathematical limits. The three primary classifications of numerical progressions are Arithmetic, Geometric, and Harmonic sequences.

## Arithmetic Progressions

An Arithmetic Progression (AP), or an arithmetic sequence, is an ordered series of numbers where the algebraic difference between any successive term and its preceding term remains a fixed numerical constant. This unchanging value is known as the common difference (d).

### 1. The $n^{th}$ Term of an Arithmetic Progression

To determine any discrete value within an arithmetic progression without manually listing every term, you can map the linear growth from the initial term. The formula for the $n^{th}$ term ($a_n$) is defined as: $a_n = a_1 + (n - 1)d$

In this expression, $a_1$ represents the first term of the sequence, n represents the total number of terms or the specific position number being evaluated, and d represents the common difference.

### 2. Sum of an Arithmetic Progression

When an engineering problem requires the accumulation or total summation (S) of the first n terms of an arithmetic progression, two distinct mathematical equations can be applied depending on what parameters are given:

- **When the first and last terms are known:** $S = \frac{n}{2}\left(a_1 + a_n\right)$
- **When the last term is unknown but the common difference is provided:**
  $S = \frac{n}{2}\left[2a_1 + (n - 1)d\right]$

### 3. Arithmetic Mean

The arithmetic mean ($A_m$), commonly referred to as the simple average, is the value obtained by summing two or more terms together and subsequently dividing that sum by the total count of those terms. For a basic two-term scenario containing values $a_1$ and $a_2$, the single intermediate arithmetic mean is modeled as: $A_m = \frac{a_1+a_2}{2}$

## Geometric Progressions

A Geometric Progression (GP), or a geometric sequence, is an ordered series of numbers where each term after the initial one is found by multiplying the immediate preceding term by a fixed, non-zero constant known as the common ratio (r). For example, in the sequence 2, 6, 18, 54, …, each term increases exponentially because the common ratio is r = 3.

### 1. The $n^{th}$ Term of a Geometric Progression

Because a geometric progression grows or decays via constant exponential scaling, the $n^{th}$ term ($a_n$) is determined using the geometric formula: $a_n = a_1 r^{n-1}$

The common ratio (r) can be found by taking the quotient of any two adjacent terms in the sequence: $r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \ldots$

### 2. Sum of a Finite Geometric Progression

Computing the total sum (S) of a finite geometric series requires selecting the correct algebraic arrangement based on the magnitude of the common ratio (r) to avoid negative signs in the denominator:

- **For a decreasing geometric series where r < 1.0:** $S = \frac{a_1(1-r^n)}{1-r}$
- **For an increasing geometric series where r > 1.0:** $S = \frac{a_1(r^n-1)}{r-1}$

### 3. Geometric Mean

The geometric mean ($G_m$) of a set of numbers is a mathematical average that measures central tendency by evaluating the product of the terms. For a sequence of n numbers, it is calculated by taking the $n^{th}$ root of the product of all those terms: $G_m = \sqrt[n]{a_1 \cdot a_2 \ldots\ldots a_n}$

#### 4. Infinite Geometric Progressions

An infinite geometric progression is an infinite series whose successive terms continue indefinitely while maintaining a constant common ratio. If the absolute value of the common ratio is strictly less than one ($|r|<1$), the terms grow progressively smaller, causing the infinite series to converge toward a finite numerical limit. The formula for the total sum (S) of an infinite, converging geometric series is expressed as: $S = \frac{a_1}{1-r}$

If the common ratio does not satisfy the condition $|r|<1$, the series diverges, and it is impossible to compute a finite sum.

## Harmonic Progressions

A Harmonic Progression (HP) is a specialized mathematical sequence formed by taking the reciprocals of the terms of an Arithmetic Progression. In other words, a sequence of terms $h_1$, $h_2$, $h_3$, …, $h_n$ constitutes a harmonic progression if and only if their mathematical reciprocals form a valid arithmetic sequence: $\frac{1}{h_1}, \frac{1}{h_2}, \frac{1}{h_3}, \ldots, \frac{1}{h_n}$

Because harmonic sequences do not possess straightforward linear common differences or exponential common ratios, there are no unique, independent direct formulas for their sums or individual terms. To solve a harmonic progression problem on a board exam, you must first convert all terms into their respective reciprocals, solve the problem using standard **Arithmetic Progression** formulas, and then take the reciprocal of the final answer to return to the harmonic scale.