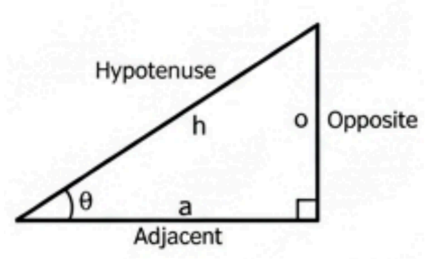
TRIGONOMETRY

Right Triangle (SOH-CAH-TOA)

	Consider a right-angled triangle with an acute angle θ. The sides of the triangle are classified relative to this angle: the **hypotenuse (h)** is the longest side and lies directly opposite the 90◦ right angle; the **opposite side (o)** is the side facing the angle θ; and the **adjacent side (a)** is the side adjacent to θ that is not the hypotenuse.

The primary and reciprocal trigonometric ratios are defined mathematically as:

$\sin\theta = \frac{o}{h}$	$\csc\theta = \frac{h}{o}$
$\cos\theta = \frac{a}{h}$	$\sec\theta = \frac{h}{a}$
$\tan\theta = \frac{o}{a}$	$\cot\theta = \frac{a}{o}$

Note: The reciprocal trigonometric functions are mathematically defined as the multiplicative inverses of the primary functions. Specifically, cosecant (csc θ) is the reciprocal of sine, secant (sec θ) is the reciprocal of cosine, and cotangent (cot θ) is the reciprocal of tangent.

The Pythagorean Theorem

The fundamental relationship between the sides of a right triangle is governed by the **Pythagorean Theorem**. This theorem states that in any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Mathematically, it is expressed as: $h^2 = o^2 + a^2$

This geometric relationship is vital for solving unknown side lengths in right triangles before computing specific trigonometric ratios.

Trigonometric Identities

1. Reciprocal and Quotient Identities

Trigonometric equations are simplified and solved using fundamental algebraic identities. Quotient identities express tangent and cotangent in terms of sine and cosine. Reciprocal identities express secant, cosecant, and cotangent as direct inverses of cosine, sine, and tangent, respectively.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\cos\theta = \frac{1}{\sec\theta}$ $\sin\theta = \frac{1}{\csc\theta}$

2. Pythagorean Relations (Pythagorean Identities)

Derived by dividing the terms of the Pythagorean Theorem ($h^2=o^2 + a^2$) by the squares of the individual sides, the Pythagorean identities are crucial for converting one trigonometric function into another.

The three primary Pythagorean relations are:

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1+\cot^2\theta = \csc^2\theta$

These identities are heavily utilized in calculus (such as trigonometric substitution in integration) and in simplifying complex algebraic expressions on the board exam.

3. Sum and Difference of Two Angles

When dealing with the sum or difference of two distinct angles, x and y, specific expansion formulas must be applied. These formulas allow us to evaluate trigonometric functions of non-standard angles by breaking them down into the sum or difference of standard, known angles (such as 30◦, 45◦, and 60◦).

Sum Identities:

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$

Difference Identities:

If considering the difference of two angles, the signs of the operators are changed.

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

4. Double-Angle Identities

Double-angle identities are special cases of the sum identities where y = x. They allow for the reduction of the angle's argument from 2x to x, which is essential when solving trigonometric equations of higher degrees.

The standard double-angle formulas are:

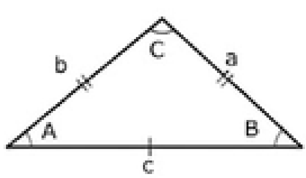
$\sin 2x = 2\sin x \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$

(Note: Utilizing the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, the cosine double-angle identity can also be expressed in the alternate forms: $\cos 2x = 2\cos^2 x - 1$ or $\cos 2x = 1 - 2\sin^2 x$.)

Oblique Triangles

	An oblique triangle is any triangle that does not contain a right angle. It can be either acute (all angles less than 90◦) or obtuse (one angle greater than 90◦). To solve for the unknown sides and angles of oblique triangles, we utilize the Law of Sines and the Law of Cosines.

5. The Law of Sines

The Law of Sines establishes that the lengths of the sides of any triangle are proportional to the sines of their opposite angles. This law is best applied when you are given:

1. Two angles and any side (AAS or ASA).
2. Two sides and an angle opposite one of them (SSA, also known as the ambiguous case).

The formula is expressed as: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

6. The Law of Cosines

The Law of Cosines is a generalization of the Pythagorean Theorem that applies to any triangle, not just right triangles. It relates the three sides of a triangle to the cosine of one of its angles. This law is applied when you are given:

1. Two sides and the included angle (SAS).
2. Three sides (SSS).

The mathematical formulation for the side a (with corresponding equations existing symmetrically for sides b and c) is: $a^2 = b^2 + c^2 - 2bc\cos A$

When given three sides (SSS), this equation can be algebraically rearranged to solve for the unknown angle A: $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

Logarithms

Logarithms are mathematical operations that serve as the inverse of exponentiation. Structurally, a logarithm determines the exponent to which a fixed number, known as the base, must be raised to produce a given value. Formally, the logarithmic statement is defined as:

$$\log_a M = N \quad \Leftrightarrow \quad a^N = M$$

In this relationship, a represents the base (where $a > 0$ and $a \neq 1$), M is the argument (which must be strictly positive, $M > 0$), and N represents the exponent. This equivalence is the single most critical tool for translating exponential equations into logarithmic forms and vice versa during board exams.

Logarithmic identities allow complex multiplicative, division, and power operations to be simplified into basic addition, subtraction, and multiplication. Mastering these properties is essential for solving algebraic equations and simplifying calculus derivatives or integrals.

1. Product Rule

The product rule states that the logarithm of a product is equal to the sum of the logarithms of its individual factors. This property allows us to break down complex multiplication into simpler addition.

- **Identity:** $\log_a(MN) = \log_a M + \log_a N$
- **Example:** $\log_3 21 = \log_3(7 \times 3) = \log_3 7 + \log_3 3$

2. Quotient Rule

The quotient rule establishes that the logarithm of a fraction is equal to the difference between the logarithm of the numerator and the logarithm of the denominator.

- **Identity:** $\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N$
- **Example:** $\log_2\left(\frac{7}{3}\right) = \log 27 - \log 23$

3. Power Rule

The power rule states that the logarithm of a number raised to an exponent is equal to the exponent multiplied by the logarithm of the number itself. This property effectively "brings down" an exponent, making it a powerful tool for solving variable-exponent equations.

- **Identity:** $\log_a(M^N) = N \log_a M$
- **Example:** $\log_2(7^3) = 3 \log_2 7$

4. Identity and Zero Properties

These basic properties establish the fundamental reference points of any logarithmic system.

- **Logarithm of the Base:** $\log_a a = 1$. This is because raising any base a to the power of 1 yields the base itself ($a^1 = a$).
    - *Example:* $\log_3 3 = 1$

- **Logarithm of One:** $\log_a 1 = 0$. This holds true because any non-zero base raised to the power of 0 is equal to 1 ($a^0 = 1$).
    - *Example:* $\log_3 1=0$

5. Inverse Property (Exponentiation of a Logarithm)

When a base is raised to a logarithm with the same base, the exponential and logarithmic operations cancel each other out, leaving only the argument of the logarithm.

- **Identity:** $b^{log_b r} = r$
- **Example:** $2^{log_2 3} = 3$

6. Change-of-Base Formula

Because standard scientific calculators typically only feature buttons for common logarithms (base 10, denoted as log) and natural logarithms (base e, denoted as ln), the change-of-base formula is necessary to evaluate logarithms with arbitrary bases.

- **Identity:** $\log_a b = \frac{log_c b}{log_c a}$ (commonly written as $\frac{log\ b}{log\ a}$ or $\frac{ln\ b}{ln\ a}$)
- **Example:** $\log_2 5 = \frac{log\ 5}{log\ 2}$ ≈2.3219

Natural Logarithms

The Constant e and Natural Log Notation

Natural logarithms are a specific class of logarithms that use the mathematical constant e as their base. The constant e, also known as Euler's number or the Naperian base, is an irrational, transcendental number approximately equal to: $e \approx 2.71828$

Instead of writing $\log_e x$, mathematicians and engineers use the notation **ln x**. Natural logarithms are fundamental to calculus, growth and decay models, and complex engineering systems.

1. Properties of Natural Logarithms

Due to the inverse nature of exponential functions with base e and natural logarithms, several simplified properties emerge:

- **Logarithm of the Base:** Since the base of the natural log is e, applying the base identity gives: $ln\, e = 1$

- **Logarithm of an Exponential:** Applying the power rule to a natural log of base e yields:
  $$ln\ e^n = n\, ln\, e = n(1) = 1$$

- **Inverse Property:** Raising the base e to the natural log of an argument x cancels out the operations: $e^{ln\, x} = x$