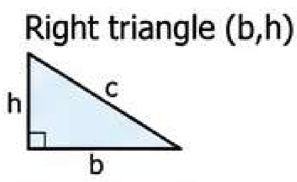
PLANE GEOMETRY

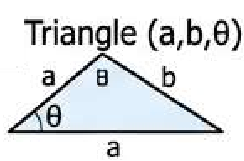
Plane Geometry is a fundamental branch of mathematics concerned with the study of two-dimensional shapes, such as lines, circles, and polygons, that are drawn on the same flat surface, mathematically referred to as a **plane**. In this domain, shapes possess only two dimensions: length and width. Consequently, calculations of volume are absent, and our focus is dedicated strictly to analyzing properties such as perimeter, diagonal lengths, and area.

The Triangle
A **triangle** is a three-sided polygon whose interior angles always sum to 180°. Depending on the specific geometric parameters provided in a given board exam question, multiple distinct formulas can be utilized to calculate its area (A_T).

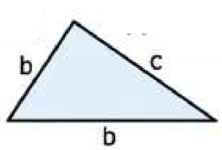
Base and Height Method

	The most basic and universally recognized method for calculating the area of a triangle is applied when the lengths of a specific base (b) and its corresponding perpendicular height (h) are known. The height represents the shortest distance from the vertex opposite the base to the line containing the base itself.	$A_T = \frac{1}{2}bh$

SAS (Side-Angle-Side) Method

	When a triangle's perpendicular height is unknown, but we are given the lengths of two adjacent sides (denoted here as a and b) and the measurement of their included angle (θ), trigonometry allows us to determine the area directly. The height of the triangle can be expressed as $h = a\,sin\theta$	$A_T = \frac{1}{2}ab\,sin\theta$

Three Sides Method (Hero's / Heron's Formula)

	When a problem provides the lengths of all three sides (a, b, and c) but supplies no angles or altitude measurements, we must employ **Hero's Formula** (often historically referred to as *Heron's Formula*).

This method relies on first computing the **semi-perimeter** (s), which represents exactly half of the triangle's total perimeter:

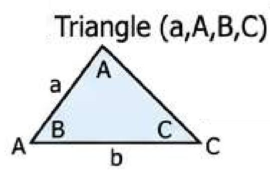
$$s = \frac{a+b+c}{2}$$

Once the semi-perimeter is calculated, it is substituted into the primary area formula:

$$A_T = \sqrt{s(s-a)(s-b)(s-c)}$$

One Side and Three Angles Method

In scenarios where we are given the values of all three interior angles (A, B, and C) alongside the length of only a single side (such as side a), the Law of Sines is used to derive a direct area formula. This eliminates the intermediate step of calculating the remaining side lengths. The mathematical formulation is:

	$A_T = \frac{a^2 \sin B \sin C}{2 \sin A}$

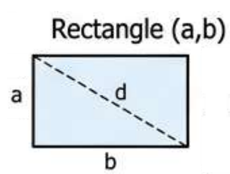
In any triangle, the side length used in the numerator's squared term (in this case, a) must always correspond to the opposite angle placed in the denominator (in this case, sin A). If you are given side b instead, the formula symmetrically adapts to $A_T = \frac{b^2 \sin A \sin C}{2 \sin B}$

Quadrilaterals

Quadrilaterals are four-sided polygons. Two of the most commonly encountered quadrilaterals on board exams are rectangles and squares, both of which feature interior angles that are all exactly 90° (right angles).

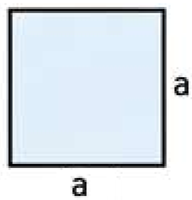
The Rectangle

A rectangle is a parallelogram containing four right angles, where opposite sides are parallel and equal in length. Let b represent the base (or length) and a represent the height (or width).

	**Area (A):** The total two-dimensional space enclosed within the boundary of a rectangle is the product of its adjacent sides	$A = ab$
	Perimeter (P): The total boundary length of the rectangle is calculated by summing all four sides.	P = 2(a + b)
	Diagonal (d): Because a diagonal splits a rectangle into two congruent right-angled triangles, we can apply the Pythagorean Theorem	$d = \sqrt{a^2 + b^2}$

The Square

A square is a highly symmetric regular quadrilateral where all four sides are equal in length (denoted as a) and all four interior angles are exactly 90°. A square is mathematically classified as both a special rectangle and a special rhombus.

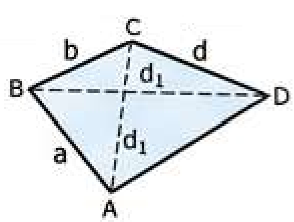
	**Area (A):** Because the base and height are equal, the area is simply the square of the side length.	$A = a^2$
	Perimeter (P): Since all four sides are of equal length, the perimeter is the sum of these four sides.	P = 4a
	Diagonal (d): Applying the Pythagorean Theorem to a square yields $d = \sqrt{a^2 + b^2}$. Simplifying this expression gives us a direct relationship	$d = a\sqrt{2}$

ADVANCED PLANE GEOMETRY

General Quadrilaterals

A quadrilateral is a four-sided polygon lying in a single two-dimensional plane. Unlike regular shapes like squares or rectangles, a general (or irregular) quadrilateral does not necessarily possess equal sides, parallel lines, or equal angles. To calculate its area, engineers must rely on specific relationships involving its diagonals, sides, and angles.

Diagonal and Included Angle (Angle of Intersection)

	If the lengths of the two diagonals (d_1 and d_2) and their acute or obtuse angle of intersection (θ) are known, the area (A) can be determined using a formula derived from vector cross-products: This formula is universally applicable to *any* quadrilateral—including self-intersecting, concave, or convex ones—as long as the lengths of the diagonals and their intersecting angle are known.	$A = \frac{1}{2} d_1 d_2 sin\theta$

Two Opposite Triangles (SAS Splitting)

When the four side lengths (a, b, c, and d) are known along with two opposite interior angles (such as A and C), the quadrilateral can be split into two distinct triangles using a diagonal. The total area is the sum of the areas of these triangles calculated via the SAS (Side-Angle-Side) method:	$A = \frac{1}{2} bc\ sinC + \frac{1}{2} ad\ sinA$

Four Sides and Opposite Angles (Bretschneider's Formula)

When all four side lengths (a, b, c, d) and the sum of two opposite angles are provided, we utilize **Bretschneider's Formula**. This formula is a generalization of Hero's Formula for four-sided polygons:

$$A = \sqrt{s(s - a)(s - b)(s - c)(s - d) - abcd\ cos^2\theta}$$

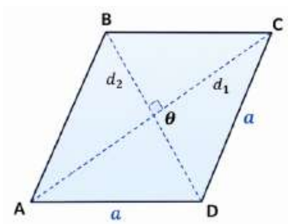
Where the semi-perimeter (s) is defined as:

$$s = \frac{a+b+c+d}{2}$$

And the angle θ is half the sum of two opposite angles:

$$\theta = \frac{1}{2}(A + C) \qquad \text{or} \qquad \theta = \frac{1}{2}(B + D)$$

The Rhombus



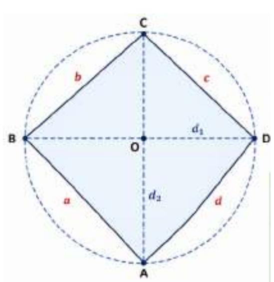
A rhombus is a special type of parallelogram where all four sides are equal in length (a). A key geometric property of a rhombus is that **its diagonals intersect at a right angle (θ = 90º)**.

Because sin(90º) = 1, the general diagonal area formula simplifies beautifully to:

$$A = \frac{1}{2}d_1 d_2$$

Cyclic Quadrilaterals

A **cyclic quadrilateral** is a four-sided polygon whose vertices all lie on a single enclosing circle (circumscribed circle).



Area (Brahmagupta's Formula)

Because the opposite angles of any cyclic quadrilateral always sum to 180º (A + C = 180º), the term θ in Bretschneider's formula becomes:

$$\theta = \frac{1}{2}(180^\circ) = 90^\circ$$

Since cos(90º) = 0, the trailing cosine term vanishes entirely. This gives us **Brahmagupta's Formula**:

$$A = \sqrt{s(s - a)(s - b)(s - c)(s - d)}$$

Radius of the Circumscribed Circle (Circumradius r)

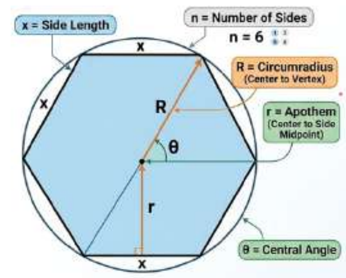
To find the radius (r) of the circle circumscribing a cyclic quadrilateral, we use Parameshvara's formula, which relates the four sides and the calculated area (A):

$$r = \frac{\sqrt{(ab+cd)(ac+bd)(ad+bc)}}{4A}$$

Regular Polygons

Classifying Polygons

- **Equilateral Polygons:** Polygons where all sides are equal in length.
- **Equiangular Polygons:** Polygons where all interior angles are equal.
- **Regular Polygons:** Polygons that are **both** equilateral and equiangular.



Key Formulas for Regular n-gons

For any regular polygon with n sides, side length x, circumradius R, and apothem (inradius) r:

Exterior Angle (θ)	The sum of the exterior angles of any convex polygon is always 360°. For a regular polygon, each exterior angle (which is also equal to the central angle subtended by one side) is:	$\theta = \frac{360^\circ}{n}$
Interior Angle	The interior angles and exterior angles at any vertex are supplementary (they sum to 180°). The measure of each interior angle is given by:	interior angle = $\frac{n-2}{n}$*180°
Perimeter (P)	The total boundary length is the product of the number of sides (n) and the length of a single side (x):	P = nx

Area (A)

The area of a regular polygon can be calculated using two primary methods depending on the parameters provided:

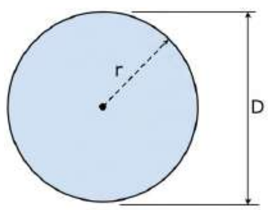
- **Using Circumradius (R):** If the distance from the center to a vertex (R) is known, the polygon can be divided into n congruent isosceles triangles, each with an area of $\frac{1}{2}R^2 \sin\theta$. The total area is:

$$A = \frac{1}{2}R^2(\sin\theta)n$$

- **Using Apothem (r) and Side Length (x):** If the apothem (the perpendicular distance r from the center to the midpoint of a side) and the side length (x) are known, the polygon is divided into n triangles with base x and height r:

$$A = \frac{1}{2}xrn$$

The Circle



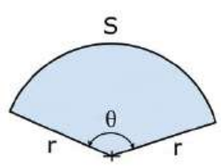
A circle is a fundamental two-dimensional locus of points situated at an equal distance, known as the radius (r), from a fixed central point. In structural and environmental engineering calculations, circular dimensions are frequently expressed in terms of the outer diameter (D), where D = 2r.

The boundary length or perimeter of a circle is called the **circumference (C)**, while the closed space within is its **area (A)**. Mathematically, these relationships are formulated as:

$$Circumference = 2\pi r = \pi D$$

Area (A) = $\pi r^2 = \frac{\pi}{4}D^2$

Sector of a Circle



A sector of a circle is a pie-shaped region bounded by two radii and an arc. The angle subtended at the center by the two radii is denoted by θ. Calculating the arc length (S) and area (A) of a sector requires working with angular measurements.

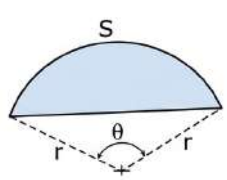
- **Arc Length (S):** The length of the curved boundary of the sector is directly proportional to the central angle when expressed in radians:

$$S = r\theta_{rad}$$

- **Area of a Sector:** The region enclosed by the sector is calculated using:

$$A = \frac{1}{2}r^2\theta_{rad}$$

Segment of a Circle



A segment of a circle is a region bounded by a chord and an arc, excluding the central vertex. Geometrically, a segment can be categorized as a minor segment or a major segment.

Minor Segment (Less than a Semicircle)

A minor segment represents the area of a sector minus the area of the isosceles triangle formed by the two boundary radii and the chord.

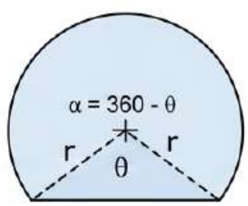
$$\text{Area (A)} = A_{Sector} - A_{Triangle}$$

Using the SAS area formula for the triangle ($A_{Triangle} = \frac{1}{2}r^2 \sin\theta$), the equation becomes:

$$A = \frac{1}{2}r^2\theta_{rad} - \frac{1}{2}r^2 \sin\theta$$

*Note: In the second term, $\sin\theta$ can be evaluated using either degree or radian mode on your calculator, provided the calculator angle setting matches your input θ. However, the first term θ_{rad} **must** be expressed strictly in radians.*

Major Segment (Greater than a Semicircle)



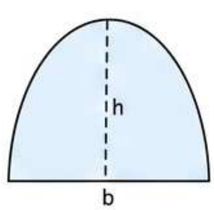
A major segment spans an angle greater than 180°. Geometrically, its area is computed by adding the area of the corresponding central triangle to the area of the major sector defined by the reflex angle $\alpha = 360° - \theta$.

$$\text{Area (A)} = A_{Sector} + A_{Triangle}$$

$$A = \frac{1}{2}r^2\alpha_{rad} + \frac{1}{2}r^2 \sin\theta$$

The Parabolic Segment

A parabolic segment is a planar region bounded by a parabola and a straight line (chord) perpendicular to its axis of symmetry. Let b represent the base width of the chord and h represent the perpendicular height (altitude) measured from the base to the vertex of the parabola.



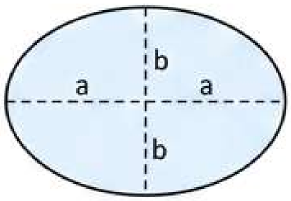
According to Archimedes' Quadrature of the Parabola, the area enclosed by a parabolic segment is exactly two-thirds of the area of a circumscribing rectangle with the same base and height:

$$\text{Area (A)} = \frac{2}{3}bh$$

Board Exam Application: This formula is highly useful in civil engineering for computing the shear and moment diagram areas of beams subjected to uniformly distributed loads, as well as in estimating earthwork volumes in highway surveying.

The Ellipse

An ellipse is a closed, symmetric curve formed by the intersection of a cone with an oblique plane. It is mathematically defined as the locus of all points where the sum of the distances to two fixed focal points (foci) is constant.



An ellipse is characterized by its semi-major axis (a, half of the longest diameter) and its semi-minor axis (b, half of the shortest diameter). The formula for its area (A) is:

Area (A) = πab

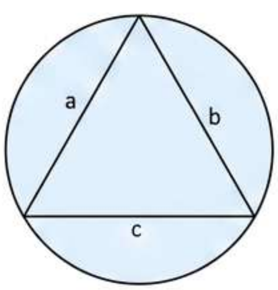
Note: If a = b = r, the ellipse becomes a perfect circle, and the formula simplifies back to the familiar circular area equation, $A = \pi r^2$

Triangles and Associated Circles

A classic category of problems in board examinations involves the interaction between a triangle and a circle. Depending on whether the circle lies outside, inside, or tangent to the extensions of the triangle's sides, we classify them as circumscribed, inscribed, or escribed circles.

Circle Circumscribed About a Triangle (Circumcircle)

A **circumscribed circle** (or **circumcircle**) is a circle that passes through all three vertices of a triangle. The center of this circle is called the **circumcenter**, which is geometrically defined as the intersection point of the perpendicular bisectors of the triangle's sides.

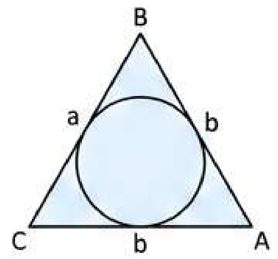


To find the radius of a circumscribed circle (commonly referred to as the **circumradius**, r), we relate the product of the three side lengths (a, b, and c) to the area of the triangle (A_T):

r = $\frac{abc}{4A_T}$

This formula is derived directly from the Law of Sines $\left(2r = \frac{a}{sinA}\right)$ and the SAS triangle area formula $\left(A_T = \frac{1}{2}bc\,sinA\right)$. On the board exam, if the three sides are known, you can compute A_T using Hero's Formula $\left(A_T = \sqrt{s(s-a)(s-b)(s-c)}\right)$ and substitute it directly into this circumradius equation.

Circle Inscribed in a Triangle (Incircle)



An **inscribed circle** (or **incircle**) is the largest possible circle that can fit entirely inside a triangle; it is tangent to all three sides. The center of this circle is the **incenter**, which is the intersection point of the triangle's interior angle bisectors.

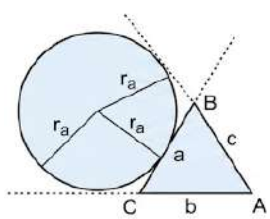
The radius of an inscribed circle (the **inradius**, r) is calculated by dividing the area of the triangle (A_T) by its semi-perimeter (s):

$$r = \frac{A_T}{s}$$

Where the semi-perimeter (s) is defined as:

$$s = \frac{a+b+c}{2}$$

Circle Escribed About a Triangle (Excircle)



An **escribed circle** (or **excircle**) is a circle lying outside the triangle that is tangent to one of the triangle's sides and to the extensions of the other two sides. Every triangle has exactly three distinct excircles, each opposite one of its three vertices.

The radius of the escribed circle opposite to side a (denoted as r_a) is calculated using the triangle's area (A_T) and its semi-perimeter (s):

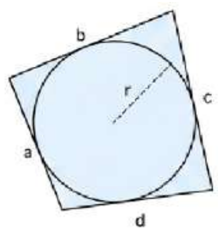
$$r_a = \frac{A_T}{s-a}$$

Symmetrically, the radii of the excircles tangent to sides b and c are given by:

$$r_b = \frac{A_T}{s-b} \qquad \text{and} \qquad r_c = \frac{A_T}{s-c}$$

Quadrilaterals and Associated Circles

Circle Inscribed in a Quadrilateral (Inscribed Circle / Tangential Quadrilateral)



When a circle is drawn such that it is tangent to all four sides of a quadrilateral, the circle is **inscribed** inside the quadrilateral. This type of quadrilateral is historically referred to as a **tangential quadrilateral**.

The radius (r) of a circle inscribed inside a quadrilateral is calculated using the quadrilateral's area (A_{quad}) and its semi-perimeter (s):

$$r = \frac{A_{quad}}{s}$$

Where the semi-perimeter (s) of the four-sided figure is defined as:

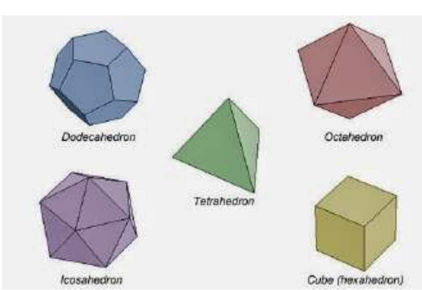
$$s = \frac{a+b+c+d}{2}$$

For a tangential quadrilateral that is *also* cyclic (known as a **bicentric quadrilateral**), the area can be calculated using the simplified formula:

$$A_{quad} = \sqrt{abcd}$$

SOLID GEOMETRY

While plane geometry deals exclusively with two-dimensional shapes resting on a flat, single plane, **solid geometry** is the branch of mathematics that treats shapes and three-dimensional figures that do not lie in the same plane. In solid geometry, objects possess three fundamental dimensions: length, width, and depth (or height). Because these figures occupy three-dimensional space, our mathematical analysis expands beyond perimeter and area to evaluate surface areas (both lateral and total) and spatial volume.

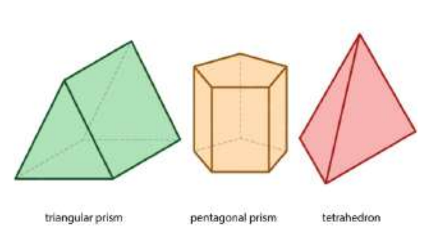


A **polyhedron** (plural: *polyhedrons* or *polyhedra*) is a three-dimensional solid bounded by flat polygonal faces, straight line segments called edges, and points where the edges meet called vertices.

The flat surfaces forming the boundary of the solid are always polygons (such as triangles, quadrilaterals, or pentagons). The straight lines where these flat faces intersect are the **edges**, and the corners where three or more edges terminate are the **vertices**.

Regular polyhedrons (often called Platonic solids) feature faces that are congruent regular polygons. The original material illustrates several key examples:

- **Tetrahedron:** A polyhedron with four triangular faces.
- **Hexahedron (Cube):** A regular polyhedron bounded by six square faces.
- **Dodecahedron:** A regular polyhedron made of twelve pentagonal faces.
- **Icosahedron:** A regular polyhedron consisting of twenty triangular faces.



A **prism** is a specific class of polyhedron characterized by having two congruent and parallel polygonal bases (referred to as the bases or ends) that are joined by lateral faces. These lateral faces are always parallelograms.

In a **right prism**, the lateral edges and faces are perpendicular to the bases, meaning the lateral faces are rectangles. In an **oblique prism**, the lateral edges are not perpendicular to the bases, making the lateral faces non-rectangular parallelograms.

Prisms are classified and named according to the shape of their base polygons:

- **Rectangular Prism:** A prism whose bases are rectangles. If all faces are rectangular, it forms a cuboid.
- **Pentagonal Prism:** A prism with two parallel pentagonal bases and five lateral faces.
- **Hexagonal Prism:** A prism with two parallel hexagonal bases and six lateral faces.

To solve exam problems involving prisms, you must master these core spatial relationships:

- **Volume (V):** The total space occupied by any prism is the product of its base area (A_{base}) and its perpendicular height (h).

  $$V = (A_{base})(h)$$

- **Lateral Surface Area (LSA):** For a right prism, the sum of the areas of all lateral faces is the perimeter of the base (P_{base}) multiplied by the height (h).

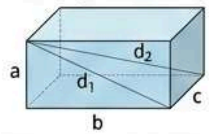
  $$LSA = (P_{base})(h)$$

- **Total Surface Area (TSA):** The total surface area of any prism is the sum of its lateral surface area and the areas of its two bases.

  $$TSA = LSA + 2A_{base}$$

PRISMS AND PYRAMIDS

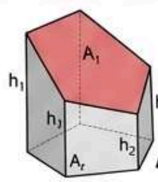
Rectangular Prism



V A_L A_S d^2

Volume	$V = abc$
Lateral	$A_L = 2(ab + ac)$
Surface	$A_S = 2(ab + bc + ac)$
Face Diagonal	$d_1 = \sqrt{a^2 + b^2}$
Space Diagonal	$d_2 = \sqrt{a^2 + b^2 + c^2}$

Truncated Prism

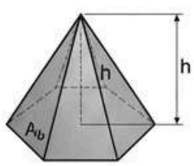


A_r (Right Section)

$$V = A_r \frac{\Sigma h}{n}$$

A_r = area of right section
n = number of sides
Σh = sum of lateral edges

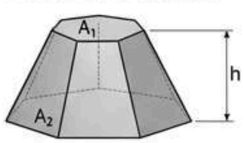
Pyramid



$$V = \frac{1}{3}Abh$$

A_b = area of the base

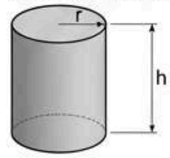
Frustum of a Pyramid



$$V = \frac{h}{3}\left(A_1 + A_2 + \sqrt{A_1A_2}\right)$$

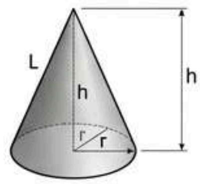
A_1, A_2 = areas of top & bottom bases

CONES, CYLINDERS, AND CURVED FRUSTUMS



$$V = \pi r^3 h$$

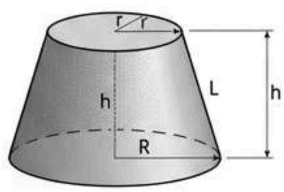
Right Circular Cone



$$V = \frac{1}{3}\pi r^2 h$$

Lateral area $A_L = \pi rL$

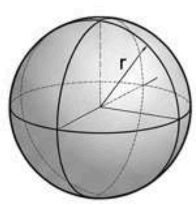
Frustum of a Right Circular Cone



$$V = \frac{\pi h}{3}(R^2 + r^2 + Rr)$$

Lateral area: $A_L = \pi(R + r)L$
L = slant height

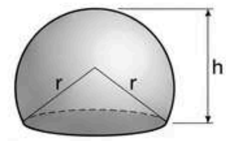
Sphere



$$V = \frac{4}{3}\pi r^3$$

SPHERICAL AND SPECIAL REVOLUTION SOLIDS

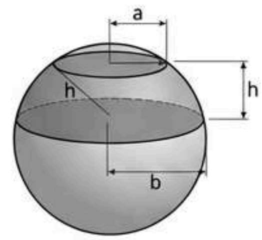
Spherical Segment of One Base



$$V = \frac{\pi h^2}{3}(3r - h)$$

h = height of cap

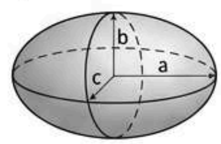
Spherical Segment of Two Bases



$$V = \frac{\pi h}{6}(3a^2 + 3b^2 + h^2)$$

a = radius of top base
b = radius of bottom base

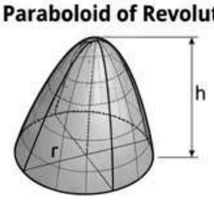
Ellipsoid



$$V = \frac{4}{3}\pi abc$$

a, b, c = semi-axes

Paraboloid of Revolution



$$V = \frac{1}{2}\pi r^2 h$$

Corrected formula based on standard derivation

Lateral area:

$$A_L = \frac{4\pi r}{3h^2}\left[\left(\frac{r^2}{4} + h^2\right)^{\frac{3}{2}} - \left(\frac{r}{2}\right)^3\right]$$