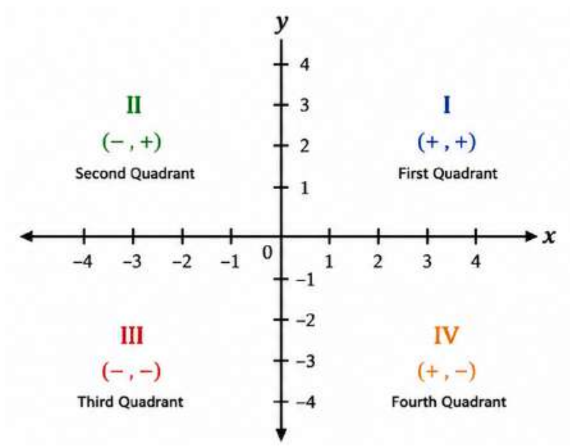
Analytic geometry, also historically referred to as coordinate or Cartesian geometry, is the mathematical discipline that bridges classical geometry with modern algebra. By establishing a systematic link between geometric shapes and algebraic equations, it allows spatial properties—such as lengths, fields, and curves—to be analyzed through numeric computation. In the context of engineering and board examinations, this framework provides the foundation for calculus, vector analysis, and structural mechanics, transforming abstract spatial problems into deterministic algebraic solutions.

The Rectangular Coordinate System

The baseline architectural framework of analytic geometry is the **Rectangular or Cartesian Coordinate System**. This system is constructed by intersecting two perpendicular real number lines at their respective zero points.



The Coordinates Axes: The horizontal reference axis is designated as the **x-axis**, while the vertical reference axis is designated as the **y-axis**. The unique point where these two reference lines intersect is defined as the **origin (O)**, which holds the baseline coordinates (0,0).
Locating Points: Every distinct physical point in this two-dimensional space is uniquely mapped as an ordered pair of numbers written in the form (x, y).

- The first value, the **x-coordinate**, is mathematically referred to as the **abscissa**. It measures the directed horizontal distance of the point relative to the y-axis.
- The second value, the **y-coordinate**, is mathematically referred to as the **ordinate**. It measures the directed vertical distance of the point relative to the x-axis.

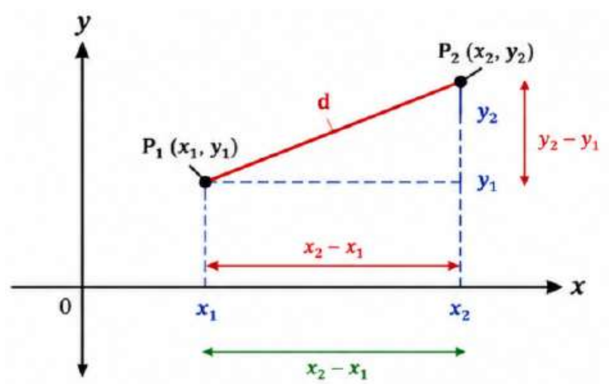
The Four Quadrants: The intersection of the two axes divides the infinite geometric plane into four distinct regions called **quadrants**, numbered counterclockwise using Roman numerals:

- **Quadrant I:** Located in the upper right region, where both coordinates are strictly positive (+, +).
- **Quadrant II:** Located in the upper left region, where the abscissa is negative and the ordinate is positive (-, +).

- **Quadrant III:** Located in the lower left region, where both coordinates are strictly negative (-, -).
- **Quadrant IV:** Located in the lower right region, where the abscissa is positive and the ordinate is negative (+, -).

The Distance Between Two Points

One of the core mathematical prerequisites for evaluating geometric relationships in a coordinate plane is finding the absolute linear separation between two points. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ represent any two distinct points plotted on the Cartesian plane.



To determine the straight-line distance (d) separating these points, we construct an implicit right-angled triangle by extending horizontal and vertical projection lines from the coordinates.

- The length of the horizontal leg is the absolute difference between the two abscissas: $\Delta x \ = \ x_2 - x_1$
- The length of the vertical leg is the absolute difference between the two ordinates: $\Delta y \ = \ y_2 - y_1$

Applying the Pythagorean theorem to this right-angled structure establishes the fundamental distance formula:

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

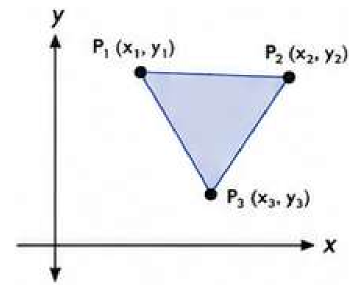
Because the directional differences are squared within the formulation, the resulting distance value will always yield a non-negative scalar quantity, regardless of whether P_1 or P_2 is designated as the initial starting point.

Area of a Triangle via Coordinates (Shoelace Formula)

In analytic geometry, calculating the area of a triangular region does not strictly require knowing its perpendicular height or base length. When a triangle is embedded within the Cartesian plane and the coordinates of its vertices are known—defined as $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$—its area (A) can be elegantly evaluated using a determinant-based method commonly known as the **Shoelace Formula** or **Gauss's Area Formula**.

The structural core of this method relies on organizing the vertex coordinates into a matrix-like array. The original material represents this calculation using a modified determinant layout:

$$A = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

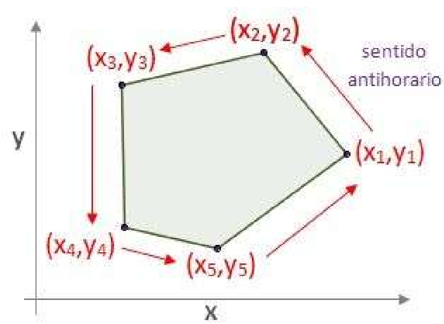


When expanded algebraically, the formula computes the absolute difference between the sum of the forward-diagonal products and the sum of the backward-diagonal products. This expansion resolves to the fundamental algebraic equation:

$$A = \frac{1}{2}\left|\left(x_1y_2 + x_2y_3 + x_3y_1\right) - \left(y_1x_2 + y_2x_3 + y_3x_1\right)\right|$$

Area of a General Polygon of n-Sides

The true power of this coordinate-based method is its scalability. It can be generalized to compute the area of any non-self-intersecting polygon (whether convex or concave) consisting of n distinct vertices labeled sequentially from P_1 to P_n. The original reference slide symbolizes this expansion with the following array structure:



$$A = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \cdots x_n \\ y_1 & y_2 & y_3 \cdots y_n \end{vmatrix}$$

To ensure this notation is functionally precise for numerical computation during your exam, it must be written in standard textbook form. The operational matrix requires you to "close" the loop of the shape by repeating the initial vertex coordinates at the very end of the array. The mathematically complete operational structure is written as follows:

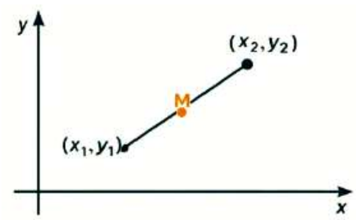
$$A = \frac{1}{2}\left|\begin{vmatrix} x_1 & x_2 & x_3 & \cdots & x_n & x_1 \\ y_1 & y_2 & y_3 & \cdots & y_n & y_1 \end{vmatrix}\right|$$

Expanding this completed matrix yields the universal polygon area equation:

$$A = \frac{1}{2}\left|\left(x_1y_2 + x_2y_3 + ... + x_ny_1\right) - \left(y_1x_2 + y_2x_3 + ... + y_nx_1\right)\right|$$

Division of a Line Segment and Midpoint Coordinates

When a straight line segment is bounded by two distinct terminal coordinates, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, any point $P(x, y)$ situated along that linear path divides the segment into a specific geometric ratio. The most foundational derivation of this property is the **midpoint formula**, which defines a point P that divides the distance between P_1 and P_2 into two perfectly equal halves.



Geometrically, this point corresponds to the arithmetic mean of the individual horizontal and vertical components of the boundary terminals. By constructing parallel projection lines to the coordinate axes, we establish proportional similar triangles along the components. Averaging the horizontal coordinates yields the abscissa of the midpoint, while averaging the vertical coordinates yields its ordinate. The coordinates of the midpoint are calculated using the following formulas:

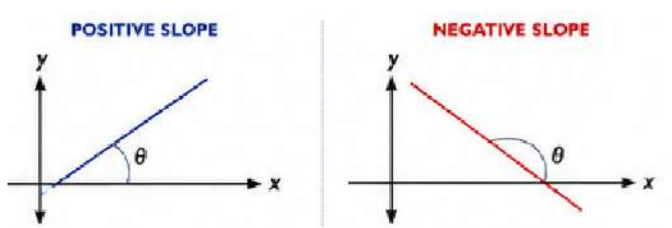
$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2}$$

Note: If a board exam question asks for a point that divides a segment into an unequal ratio (e.g., $r_1 : r_2$ or a fractional distance k from P_1), the general section formulas must be utilized:

$$x = x_1 + k(x_2 - x_1) \quad \text{and} \quad y = y_1 + k(y_2 - y_1)$$

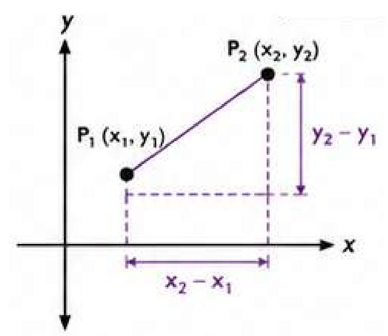
Slope and Angle of Inclination

The orientation of an infinite straight line traversing a two-dimensional plane is defined by its steepness and directional path. The geometric measure of this orientation is characterized by two distinct yet interconnected properties: the **angle of inclination** and the **slope**.



- **The Angle of Inclination (θ):** The inclination of a line is defined as the counterclockwise angle θ measured starting from the positive direction of the horizontal x-axis up to the line. The range of this angle is strictly bounded within the interval 0º ≤ θ < 180º. An inclination where 0º < θ < 90º indicates a line rising from left to right, whereas an inclination where 90º < θ < 180º indicates a falling line.
- **The Slope (m):** The slope m represents a constant numerical ratio describing the rate of vertical change relative to horizontal change (conventionally stated as the "rise over run"). Mathematically, the slope is defined as the trigonometric tangent of the line's angle of inclination:

$$m = tan\,\theta$$

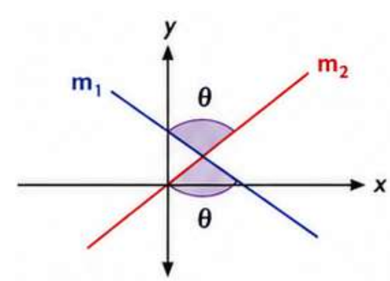


When a line passes through two known coordinates, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, a right triangle can be constructed underneath the segment path. The vertical leg represents the change in height $(y_2 - y_1)$, and the horizontal leg represents the change in span $(x_2 - x_1)$. Applying the definition of the tangent function to this triangle yields the standard coordinate formula for slope:

$$m = tan\,\theta = \frac{y_2 - y_1}{x_2 - x_1}$$

The Angle Between Two Intersecting Lines

When two lines, L_1 and L_2, intersect within a coordinate plane, they form an angle θ at their junction point. Let m_1 represent the slope of the initial line (L_1) with an inclination angle of α_1, and let m_2 represent the slope of the terminal line (L_2) with an inclination angle of α_2.



Using the exterior angle theorem from classical plane geometry, the angle of intersection can be related directly to the individual inclination angles of the two lines, establishing that $\theta = \alpha_2 - \alpha_1$. By applying the trigonometric identity for the tangent of a difference between two angles, $tan\left(\alpha_2 - \alpha_1\right) = \frac{tan\,\alpha_2 - tan\,\alpha_1}{1 + tan\,\alpha_2 tan\,\alpha_1}$, we can substitute the slopes (m = tan α) to derive the analytical angle formulation:

$$tan\,\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

Note: When calculating the angle between two lines, the order of the slopes in the numerator is highly critical. The slope m_2 must correspond to the line with the larger inclination angle (the terminal line when moving counterclockwise through the angle θ). If your calculation yields a negative value, it simply means you computed the **obtuse** angle of intersection rather than the **acute** angle. To find the acute angle, take the absolute value of the expression.

Zero Denominator Conditions: Be prepared for special slope behaviors during your board exam. If a line is perfectly vertical, its inclination is $\theta = 90°$, causing $x_2 - x_1 = 0$; its slope is mathematically **undefined**. Conversely, if two lines are perpendicular, the denominator of the intersection formula becomes zero ($1 + m_1 m_2 = 0$), which mathematically confirms the classic engineering relationship: $m_1 m_2 = -1$

Straight Lines

In analytic geometry, any straight line traversing a two-dimensional Cartesian plane can be mathematically defined by a first-degree polynomial equation in terms of x and y. The baseline mathematical architecture for this relationship is the **General Form** (or standard algebraic form) of a linear equation, expressed as:

$$Ax + By + C = 0$$

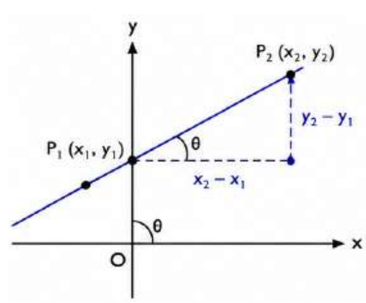
In this comprehensive equation, A, B, and C represent real number constants. To ensure that the expression represents a valid, infinite straight line rather than an empty set or a mathematical contradiction, a strict algebraic constraint must be enforced: **A and B cannot both be zero simultaneously**. If both coefficients were to vanish, the spatial variable tracking would completely collapse. Depending on the values assigned to these constants, the line can assume various structural orientations, which are systematically captured through specialized operational forms.

A. Point-Slope Form

The **Point-Slope Form** serves as the primary bridge between the purely geometric properties of a line and its algebraic equation. This form is deployed when you are given a single distinct point anchored in space, designated as $P_1(x_1, y_1)$, and the specific numerical slope (m) indicating the line's steepness.

This formulation is directly derived from the core definition of a slope. If we choose any arbitrary, moving variable point P(x, y) situated along the infinite path of the line, the constant slope m separating it from our fixed coordinate $P_1(x_1, y_1)$ must satisfy the standard rise-over-run calculation:

$$m = \frac{y - y_1}{x - x_1}$$

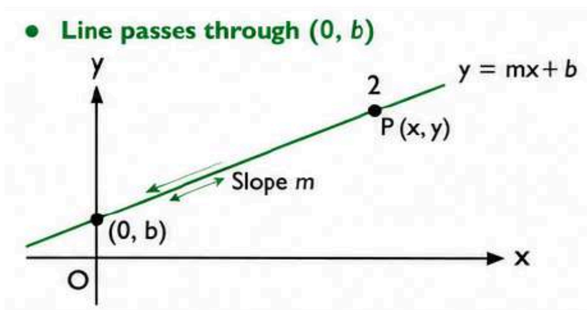


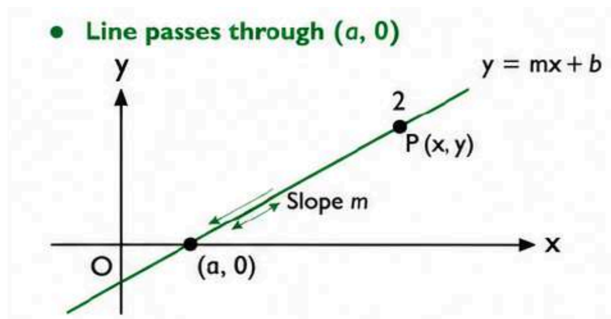
By isolating the vertical coordinate change via algebraic cross-multiplication, we arrive at the standard textbook equation for the point-slope form:

$$y - y_1 = m(x - x_1)$$

B. Slope-Intercept Form

When the specific geometric anchoring point happens to lie directly along the vertical coordinate axis, the general equation simplifies into the highly efficient **Slope-Intercept Form**. The exact point where an infinite line crosses the vertical y-axis is defined as the **y-intercept**, denoted by the coordinate (0, b), where b represents the directed vertical distance from the origin.





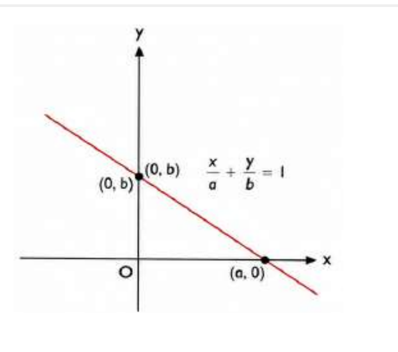
Substituting this specific intercept point (0, b) into the previously established point-slope equation gives us y - b = m(x - 0). When isolated for the vertical dependent variable, it yields the classic functional relationship:

$$y = mx + b$$

Conversely, if the line crosses the horizontal x-axis, that location is designated as the **x-intercept**, algebraically mapped at the coordinate location (a, 0), where a serves as the directed horizontal distance from the origin.

C. Intercept Form

When a line cuts diagonally across both coordinate axes without passing through the absolute origin, it simultaneously generates both an x-intercept at (a, 0) and a y-intercept at (0, b). The spatial relationship can be modeled using the highly symmetrical **Intercept Form**.



This form cleanly organizes the geometric parameters by dividing each coordinate variable by its respective directional axis intercept. This structured arrangement ensures that the sum of the fractional intercepts evaluates to unity:

$$\frac{x}{a} + \frac{x}{b} = 1$$

Note: The intercept form carries strict algebraic limitations that frequently appear in multiple-choice traps. This equation is completely invalid if the line passes directly through the origin (0,0), because dividing by zero (a=0 or b=0) is mathematically undefined. Additionally, it cannot be used to represent perfectly horizontal or vertical lines, as these configurations run parallel to an axis and never produce a matching second intercept.

Geometric Relationships Between Two Lines

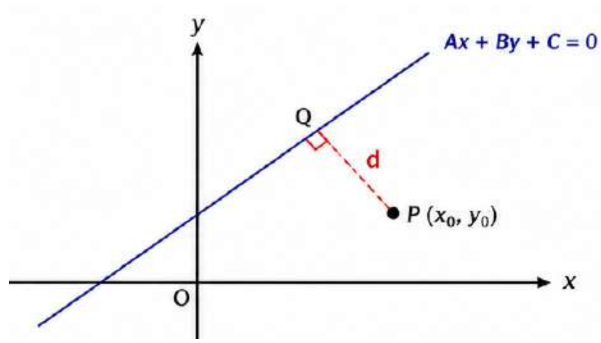
When evaluating the spatial interaction between two separate lines, L_1 and L_2, with determined slopes designated as m_1 and m_2, their relative orientation falls under two major categories:

Parallelism: Two lines are defined as strictly **parallel** if they lie within the same flat plane but never intersect, maintaining a perfectly uniform separation distance across their entire infinite path. For this condition to hold, their spatial steepness must be identical, leading to the direct slope identity: $m_1 = m_2$

Perpendicularity: Two lines are defined as **perpendicular** (or orthogonal) if they intersect at a single junction point at a perfect right angle (90º). Mathematically, this structural condition dictates that the slope of one line must be the negative reciprocal of the other: $m_1 m_2 = -1$

The Perpendicular Distance Between a Point and a Line

In many analytical geometry applications, it is necessary to compute the geometric separation between a point and a line. The distance d from an external point $P_1(x_1, y_1)$ to an infinite straight line defined by the general linear equation Ax + By + C = 0 is defined explicitly as the shortest possible distance between them. Geometrically, this shortest path is always measured along a line segment drawn from P_1 that intersects the target line at a perfect right angle (90º).



To establish this relationship algebraically, the line equation must be arranged into its standard general form, shifting all terms to the left member so it equates to zero. The coordinates of the external point (x_1, y_1) are then substituted directly into the linear expression. The resulting scalar is divided by the geometric magnitude of the line's directional coefficients, yielding the formal mathematical formula:

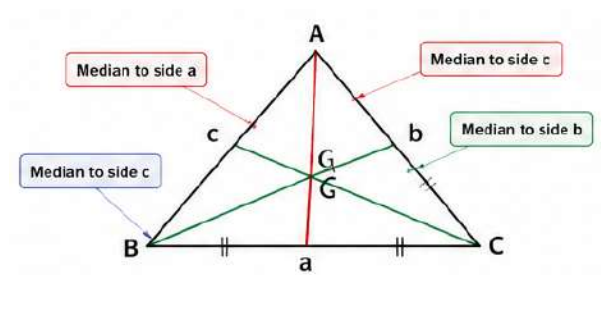
$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

In this formulation, A represents the numerical coefficient of x, B represents the numerical coefficient of y, and C represents the constant term positioned on the left side of the equation.

Elements and Properties of a Triangle

The Median and the Centroid

A **median** of a triangle is a fundamental internal Cevian defined as a line segment that connects any single vertex directly to the exact midpoint of its opposite side. Every triangle contains exactly three distinct medians, each originating from one of the three vertices (A, B, or C) and extending to the corresponding opposite boundary side (a, b, or c).



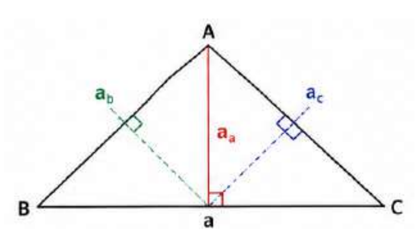
The length of a median can be determined algebraically from the lengths of the three sides of the triangle. Apollonius's theorem yields the analytical formulation for the length of the median directed to side a (denoted as m_a):

$$m_a = \sqrt{\frac{2b^2 + 2c^2 - a^2}{4}}$$

When all three medians are constructed simultaneously within the plane of the triangle, they intersect at a single, shared point of concurrency known as the **centroid**. In physical mechanics and engineering analytics, the centroid represents the true center of gravity or geometric center of a uniform triangular lamina.

The Altitude and the Orthocenter

An **altitude** of a triangle is an internal or external line segment drawn from a vertex that intersects the line containing the opposite side at a perfect right angle (90º), establishing a perpendicular relationship. The altitude serves as the true geometric height utilized in standard metric area calculations.



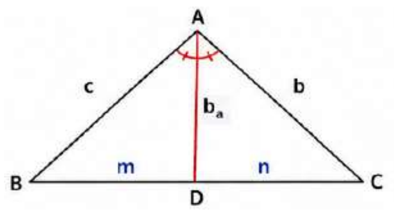
Because the basic area of a triangle (A_T) is defined as half the product of its base and its perpendicular height $\left(A_T = \frac{1}{2} base \cdot height\right)$, the length of the altitude directed to side a (denoted as a_a) can be isolated through basic algebraic rearrangement:

$$a_a = \frac{2A_T}{a}$$

When all three perpendicular altitudes are drawn, their lines of action intersect at a single concurrent intersection point designated as the **orthocenter**. Unlike the centroid, which always resides strictly inside the perimeter of the triangle, the orthocenter changes its spatial position based on the geometry of the triangle: it rests inside an acute triangle, lies exactly on the vertex of the right angle in a right triangle, and falls completely outside the perimeter in an obtuse triangle.

The Angle Bisector and the Incenter

An **angle bisector** of a triangle is an interior line segment originating from a vertex that divides the internal angle at that vertex into two perfectly equal angles. The segment extends across the interior space to terminate directly on the opposite boundary side, bisecting the specific angle included between the two adjacent sides.



The length of the angle bisector originating from vertex A and directed to side a (denoted as b_a) is computed using the semi-perimeter (s) and the adjacent side lengths (b and c):

$$b_a = \frac{2}{b+c}\sqrt{bcs(s-a)}$$

Where the semi-perimeter constant (s) represents exactly half of the total boundary perimeter:

$$s = \frac{a+b+c}{2}$$

The three internal angle bisectors converge concurrently at a single interior point known as the **incenter**. The incenter serves as the center of the unique inscribed circle (the incircle) that fits perfectly inside the triangle, touching all three boundary sides tangentially. Because it is equidistant from all three sides, it remains an essential reference point for geometric layout optimization problems.