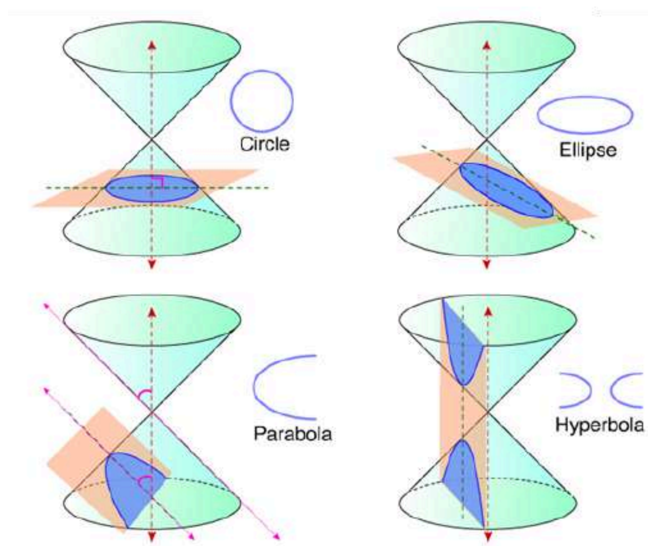
## Conic Sections

A **conic section** (or simply conic) is a curve obtained by intersecting a right circular cone with a flat plane. Depending on the angle of the intersecting plane relative to the axis and slant generator of the cone, four distinct geometric curves can be produced: the circle, the ellipse, the parabola, and the hyperbola. In the study of analytic geometry, these spatial curves are modeled algebraically using second-degree equations.

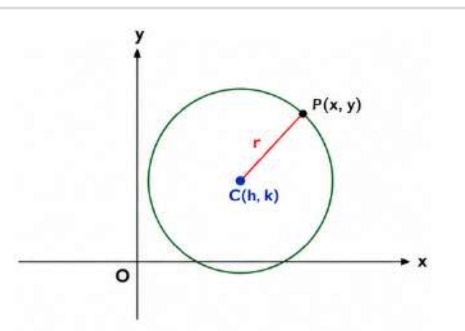



The standard mathematical framework that maps every possible conic configuration in a two-dimensional Cartesian plane is the **General Equation of Conics**, expressed as:

$$Ax^2 + By^2 + Dx + Ey + F = 0$$

In this universal expression, A, B, D, E, and F represent real constant coefficients. The specific relationships, signs, and relative magnitudes between the quadratic coefficients (A and B) dictate the geometric classification of the curve. When analyzing these equations for board examinations, recognizing how these constants govern the shape of the graph is essential for rapid problem solving.

## The Circle




A **circle** is geometrically defined as the locus or set of all points P(x, y) in a two-dimensional plane that lie at a fixed, unchanging distance from a fixed central reference point. The constant distance separating the boundary from the interior reference point is designated as the **radius** (r), while the fixed central point is defined as the **center**, traditionally mapped at the coordinate location C(h, k).

This geometric relationship can be converted into an algebraic expression by applying the distance formula between the center C(h, k) and any arbitrary variable point P(x, y) on the perimeter. Squaring both sides of the distance equation eliminates the radical, establishing the **Standard Equation of a Circle**:

$$(x - h)^2 + (y - k)^2 = r^2$$

In this standard layout, the directional translations (h, k) explicitly expose the spatial location of the center, and the isolated scalar on the opposite side directly yields the square of the radius ($r^2$). If the center of the circle resides perfectly at the absolute origin (0, 0), the standard equation simplifies directly to its basic form: $x^2 + y^2 = r^2$.

**The General Equation of a Circle**

When the binomial squared terms of the standard equation are completely expanded algebraically $(x^2 - 2hx + h^2) + (y^2 - 2ky + k^2) = r^2$ and rearranged so that all terms collect on the left-hand side, the expression matches the general second-degree conic form. For a circle, the quadratic coefficients must be perfectly equal (A = B). Dividing the entire equation by this common coefficient yields the **General Equation of a Circle**:

$$x^2 + y^2 + Dx + Ey + F = 0$$

Here, D, E, and F serve as arbitrary real constants. This general form hides the direct geometric measurements of the circle. To find the center (h, k) and radius (r) from this layout, you can complete the square for both the x and y variables groups, which establishes the following direct relationships:

$$h = -\frac{D}{2}\ ;\ k = -\frac{E}{2}\ ;\ \text{and}\ r = \sqrt{h^2 + k^2 - F}$$

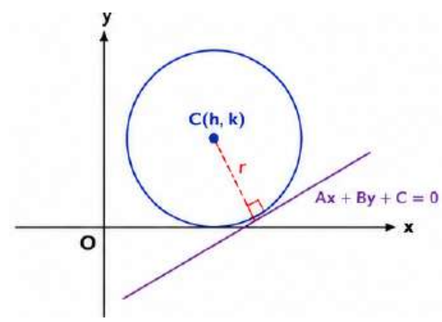
**Nature of the Constant $r^2$ (Existence Criteria)**

During algebraic transformation from the general form to the standard form, the value calculated for the isolated constant terminal on the right member ($r^2 = h^2 + k^2 - F$) determines the physical existence and geometric nature of the graph:

- **Real Circle ($r^2 > 0$):** When the evaluated constant is strictly greater than zero, the radius is a real positive number, and the graph forms a valid, two-dimensional circle in the coordinate plane.
- **Point Circle ($r^2 = 0$):** When the constant evaluates to exactly zero, the radius collapses to a length of zero. The geometric graph degenerates from a two-dimensional region down to a **single point** located exactly at the coordinate center (h, k).
- **Imaginary Circle ($r^2 < 0$):** When the constant evaluates to a negative value, taking the square root requires imaginary numbers ($r = \sqrt{negative\ value}$). Because no real coordinate pairs can satisfy a negative squared sum, there is **no real graph** produced on the physical Cartesian plane.

### Circle Tangent to a Line

A classic high-yield board exam scenario involves determining the properties of a circle that is perfectly **tangent** to an external straight line defined by the general linear equation Ax + By + C = 0. By definition, a tangent line touches the perimeter of the circle at exactly one distinct point without cutting through its interior.

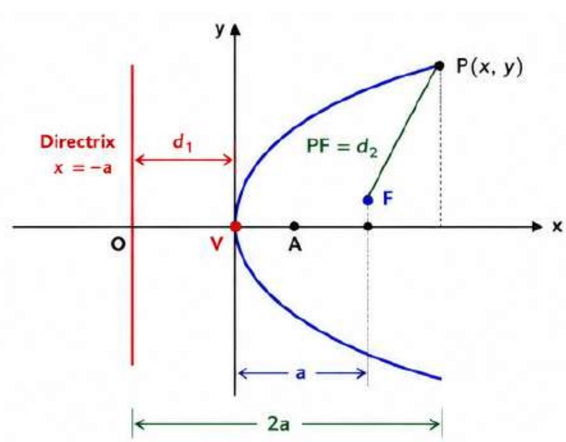



A crucial geometric property of circles dictates that a radius drawn from the center C(h, k) to the exact point of tangency will always intersect the tangent line at a perfect right angle (90°). Consequently, the radius (r) of the circle is exactly equal to the perpendicular distance from the center point to the line. Substituting the center coordinates (h, k) into the point-line distance formula yields the radius equation:

$$r = \frac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$$

## The Parabola

Following the study of the circle, the next conic section generated by specific intersecting planes is the **parabola**. Geometrically, a parabola is defined as the locus or set of a moving point whose distance from a fixed point called the **focus** (F) is perfectly equal to its perpendicular distance from a fixed line called the **directrix**.

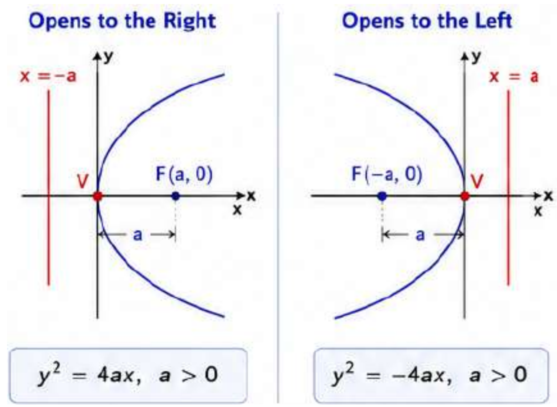



This equal-distance relationship means that the geometric **eccentricity** (e) of every parabola is constant and always equal to unity (e = 1). The structural midpoint of the segment running perpendicularly from the focus to the directrix is the **vertex** (v), which serves as the turning point of the curve. The distance from the vertex to the focus is designated as the scalar a, which by definition also equals the absolute distance from the vertex to the directrix line. The chord passing through the focus perpendicular to the axis of symmetry is the **latus rectum**, and its total length is always equal to 4a.

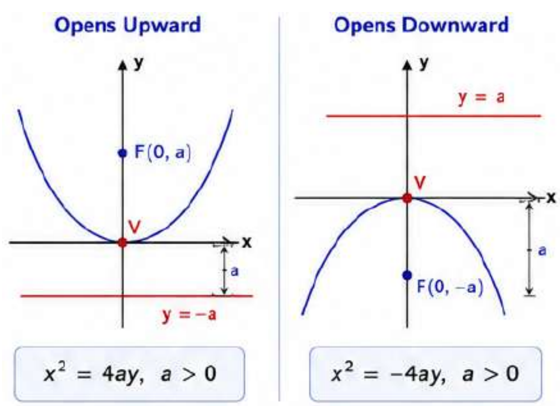
### Standard Equations of a Parabola

The orientation of a parabola is determined by whether its directrix line is situated vertically or horizontally. When the vertex is translated away from the origin to a general coordinate point V(h, k), the standard equations take two distinct forms based on spatial orientation:

- **Vertical Directrix (Horizontal Axis of Symmetry):** If the directrix line is vertical, the parabola opens either to the right or to the left. The variable y is squared, yielding the standard formula: $(y - k)^2 = 4a(x - h)$




- **Horizontal Directrix (Vertical Axis of Symmetry):** If the directrix line is horizontal, the curve opens either upward or downward. The variable  is squared, yielding the standard formula: $(x - h)^2 = 4a(y - k)$




The directional opening depends entirely on the sign of the focal parameter 4a. For a horizontal axis, a positive value (+4a) causes the curve to open toward the right, whereas a negative value (-4a) causes it to open toward the left. For a vertical axis, a positive value opens upward, and a negative value opens downward.

### General Equations of a Parabola

When the binomial squared terms of the standard equations are fully expanded and rearranged so all terms collect on one side of the equality, they form a specific second-degree conic expression where only one of the quadratic variables survives. This produces two distinct general geometric forms:

- **Horizontal Directrix Form:**

  $x^2 + Dx + Ey + F = 0$

This expression represents a parabola with a vertical axis of symmetry. Because the x term carries the quadratic exponent, the curve opens either upward or downward depending on the signs of the constants.

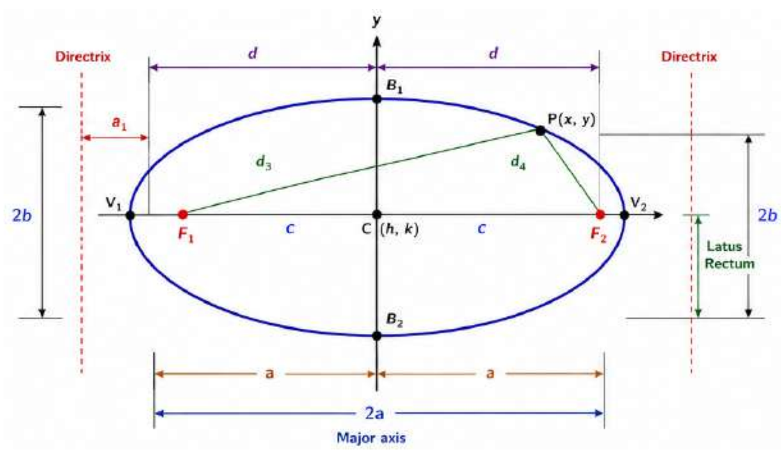
- **Vertical Directrix Form:**

  $$y^2 + Dx + Ey + F = 0$$

  This expression represents a parabola with a horizontal axis of symmetry. Because the y term carries the quadratic exponent, the curve opens either toward the right or toward the left.

## The Ellipse

Continuing the study of central conics, an **ellipse** is geometrically defined as the locus or set of all points in a two-dimensional plane such that the sum of the distances from each point on the curve to two fixed interior reference points remains constant. These two fixed internal points are designated as the **foci** ($F_1$ and $F_2$). The infinite straight line passing directly through both foci establishes the principal axis of symmetry, known as the **major axis**.




The points where the ellipse intersects its major axis are the **vertices** ($v_1$ and $v_2$). The structural midpoint of the line segment connecting these vertices is the **center** C(h, k). The total length of the major axis is defined as 2a, which means the distance from the center to either vertex is exactly equal to the semi-major axis length, a. By definition, the constant sum of the distances from any variable point on the curve to the two foci ($d_1 + d_2$) is perfectly equal to this major axis length (2a).

Perpendicular to the major axis and passing directly through the center is the **minor axis**, which has a total length of 2b, where b represents the semi-minor axis length. The distance from the center to either focal point is denoted as c. These three foundational scalar dimensions are strictly governed by the Pythagorean relationship:

$$a^2 = b^2 + c^2$$

The degree of flattening or elongation of the ellipse is quantified by its geometric **eccentricity** (e). It is defined as the ratio of the focal distance to the semi-major axis length $\left(e = \frac{c}{a}\right)$, or dynamically via focal tracking coordinates $\left(e = \frac{d_3}{d_4}\right)$. Because the foci must always reside within the interior of the vertices, the eccentricity of an ellipse is strictly bounded between zero and unity (0 < e < 1).

The ellipse also possesses two linear boundary lines known as **directrices**, located at an absolute distance of $d = \frac{a}{e}$ from the center point. Finally, the chords passing through either focus perpendicular to the major axis are the **latera recta** (plural of latus rectum); the total length of each latus rectum (LR) is given by the formula: $LR = \frac{2b^2}{a}$

### Standard Equations of an Ellipse

When the center of the ellipse is translated away from the origin to a general coordinate center C(h, k), its standard orientation determines how its denominators are allocated. Because a > b by definition in an ellipse, the larger denominator always dictates the direction of the major axis:

- **Major Axis Parallel to the x-axis (Horizontal Ellipse):** When the major axis stretches horizontally, the larger scalar parameter $a^2$ is positioned underneath the x variable expression:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

- **Major Axis Parallel to the y-axis (Vertical Ellipse):** When the major axis stretches vertically, the larger scalar parameter $a^2$ must reside underneath the y variable expression:

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

### General Equation and Existence Criteria

When the standard fractions are cleared and the squared terms expanded, they form the general second-degree conic expression. For an ellipse, the quadratic coefficients A and B must have the exact same algebraic sign, but distinct numerical magnitudes. The **General Equation of an Ellipse** is written as:

$Ax^2 + By^2 + Dx + Ey + F = 0$ where: A > 0 and B > 0

To evaluate the mathematical existence and nature of the graph from this general form without performing a full algebraic completion of the square, you can compute a specific determinant-like metric constant, M:

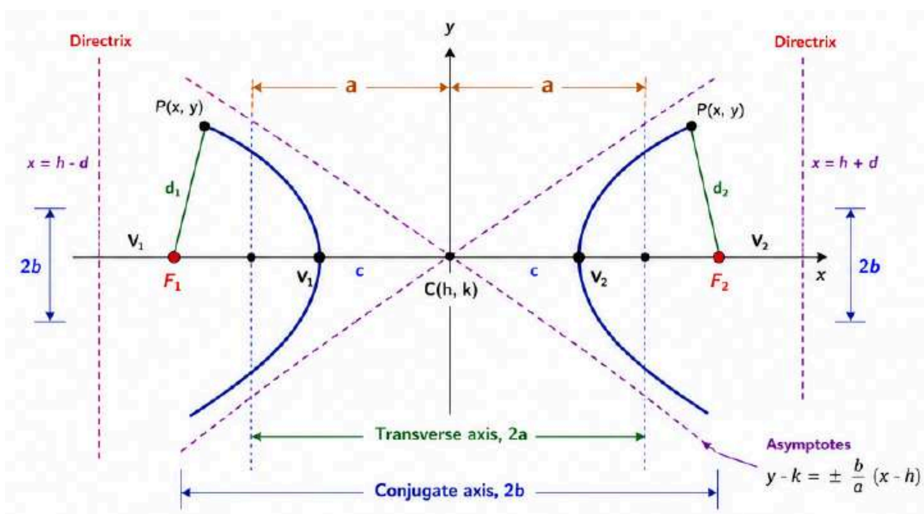
$$M = -F + \frac{D^2}{4A} + \frac{E^2}{4B}$$

The value of this metric scalar determines the geometric outcome on the Cartesian plane:

- **Real Ellipse (M > 0):** The constant is strictly positive, yielding a valid, two-dimensional curve.
- **Point Ellipse (M = 0):** The system collapses down to a single, degenerate coordinate point at the center (h, k).
- **Imaginary Ellipse (M < 0):** The system requires imaginary values to satisfy the equation, resulting in no physical graph on the real plane.

## The Hyperbola

Completing the core progression of central conics is the **hyperbola**. Geometrically, a hyperbola is defined as the locus or set of all points in a two-dimensional plane such that the absolute difference of the distances from any point on the curve to two fixed interior reference points remains constant. These two fixed internal points are designated as the **foci** ($F_1$ and $F_2$). The infinite straight line running directly through both foci establishes the principal symmetry axis, known as the **transverse axis**.




The points where the hyperbola's dual symmetric branches intersect the transverse axis are the **vertices** ($v_1$ and $v_2$). The structural midpoint of the line segment connecting these vertices is the **center**, traditionally mapped at the coordinate location C(h, k). The total length of the transverse axis (TA) is defined as 2a, meaning the distance from the center to either vertex is exactly equal to the semi-transverse axis length, a. By definition, the constant absolute difference between the distances from a variable point on the curve to the two foci is perfectly equal to this transverse axis length (2a).

Perpendicular to the transverse axis and passing directly through the center is the **conjugate axis**, which has a total length (CA) of 2b, where b represents the semi-conjugate axis length. The total length of the focal axis (FA) stretching between the two foci is designated as 2c, making c the distance from the center to either focal point. Because the foci reside completely on the exterior of the vertices in a hyperbola, these three foundational scalar dimensions are strictly governed by a hyperbolic Pythagorean relationship where c is the dominant parameter:

$$c^2 = a^2 + b^2$$

The degree of opening or divergence of the hyperbolic branches is quantified by its geometric **eccentricity** (e). It is defined as the ratio of the focal distance to the semi-transverse axis length $\left(e = \frac{c}{a}\right)$, or dynamically via focal tracking coordinates $\left(e = \frac{d_3}{d_4}\right)$. Because c > a, the eccentricity of a hyperbola is strictly greater than unity (e > 1). The hyperbola also possesses two linear boundary lines known as **directrices**, located at an absolute distance of $d = \frac{a}{e}$ from the center point. The chords passing through either focus perpendicular to the transverse axis are the **latera recta**, where the total length of each latus rectum (LR) is given by the formula: $LR = \frac{2b^2}{a}$

### Standard Equations of a Hyperbola

When the center of the hyperbola is translated away from the origin to a general coordinate center C(h, k), its standard orientation determines which variable term leads to the subtraction. Unlike the ellipse, the denominator $a^2$ is not chosen by size; it is always the denominator of the positive, leading fractional term:

- **Transverse Axis Parallel to the x-axis (Horizontal Hyperbola):** When the transverse axis stretches horizontally, the hyperbola opens to the left and right. The x-variable term is positive and leads the equation:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

- **Transverse Axis Parallel to the y-axis (Vertical Hyperbola):** When the transverse axis stretches vertically, the hyperbola opens upward and downward. The y-variable term is positive and leads the equation:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

### Equations of the Asymptotes

An **asymptote** of a curve is a straight line configured such that the perpendicular distance from the line to the curve approaches zero as the curve extends indefinitely far from the origin. Every hyperbola possesses two distinct linear asymptotes that intersect exactly at the center point C(h, k), acting as geometric structural guidelines that the branches approach but never cross.

The universal point-slope formulation for these linear asymptotes is written as:

$$y - k = \pm m(x - h)$$

Where (h, k) represents the center coordinates of the hyperbola and m represents the absolute slope value of the lines. The calculation of the slope parameter m depends directly on the spatial orientation of the hyperbola's axes:

- **Horizontal Axis:** The slope is calculated as $m = \frac{b}{a}$
- **Vertical Axis:** The slope is calculated as $m = \frac{a}{b}$

When utilizing this equation, the positive sign (+) generates the upward-sloping asymptote, while the negative sign (-) generates the downward-sloping asymptote.

### General Equation of a Hyperbola

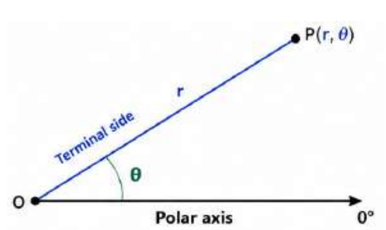
When the standard equations are cleared of denominators and expanded into polynomial form, they collapse into the general second-degree conic expression. For a hyperbola, the quadratic coefficients A and B must carry completely opposite algebraic signs, meaning one of them must be mathematically less than zero (A < 0 or B < 0). The **General Equation of a Hyperbola** is expressed as:

$$Ax^2 + By^2 + Dx + Ey + F = 0 \qquad \text{where A or B is less than 0}$$

## Polar Coordinates

While the rectangular coordinate system relies on a grid of orthogonal lines to locate a point, the **Polar Coordinate System** determines spatial positioning using a distance and an angle relative to a fixed reference framework. This system is constructed upon a single point of origin known as the **pole** (O) and an initial horizontal ray extending to the right, designated as the **polar axis**.

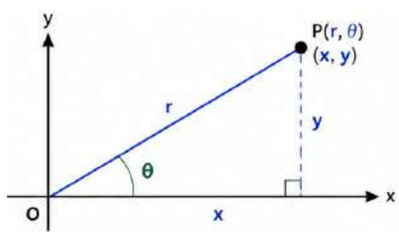
Every distinct physical point P within this two-dimensional space is uniquely mapped as an ordered pair written in the form (r, θ).




- The first parameter, r, is mathematically referred to as the **radius vector** or **modulus**. It measures the straight-line directed distance from the pole to the point P.
- The second parameter, θ, is formally defined as the **amplitude**, **argument**, or **vectorial angle**. It specifies the directed angle measured from the initial polar axis to the terminal side containing the radius vector. By geometric convention, an angle measured in a counterclockwise direction is designated as positive, whereas an angle measured clockwise is designated as negative.

**Relationship Between Rectangular and Polar Coordinates**

To establish a bridge between the rectangular Cartesian plane and the polar plane, the two systems are superimposed so that the Cartesian origin (0,0) aligns perfectly with the polar pole (O), and the positive x-axis lies directly on top of the polar axis. When an arbitrary point P is plotted, its position can simultaneously be described by the Cartesian coordinates (x, y) and the polar coordinates (r, θ).



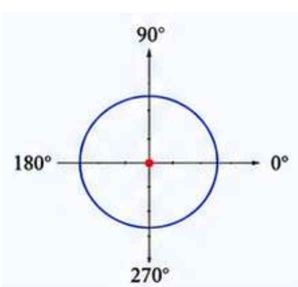

Dropping a perpendicular line from the point P to the horizontal axis forms a right-angled triangle. The hypotenuse represents the radius vector (r), the adjacent base represents the horizontal distance (x), and the vertical side represents the vertical distance (y). Applying baseline right-triangle trigonometry yields the fundamental transformation equations used to convert polar parameters into rectangular coordinates:

$x = r\cos\theta$ ; $y = r\sin\theta$

Conversely, when converting rectangular coordinates back into polar coordinates, the relationships are governed by the Pythagorean theorem and the inverse tangent function. The algebraic expressions to isolate the modulus (r) and the vectorial angle (θ) are written as follows: $r = \sqrt{x^2 + y^2}$ ; $\theta = tan^{-1}\left(\frac{y}{x}\right)$

# Curve Tracing on Polar Coordinates

When analyzing geometric paths within a polar framework, certain equations map out fundamental algebraic curves that regularly appear in spatial engineering problems. The most basic of these forms is the expression where the radius vector remains entirely independent of the vectorial angle, defined by the relation: r = k

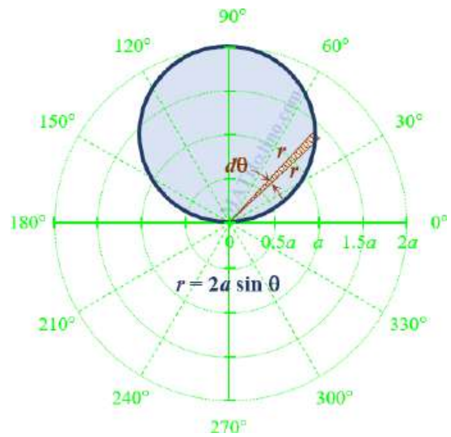


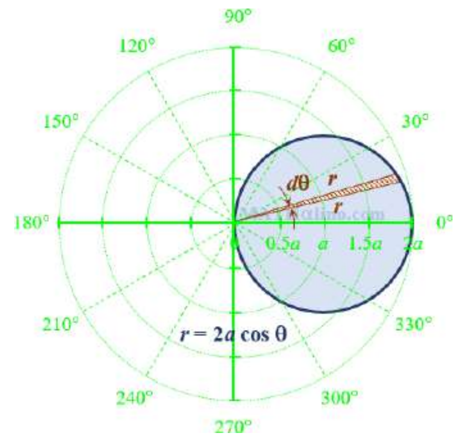

In this equation, k represents a strictly positive real constant (k > 0). Because the distance from the origin remains fixed at k units regardless of the angle θ, tracing this path generates a **circle** centered precisely at the pole.

Beyond circles centered at the origin, polar formulations can also represent circular paths that pass directly through the pole while sitting flush against one of the principal coordinate axes. These configurations are governed by the following equations:

$r = 2a\sin\theta$ and $r = 2a\cos\theta$

For these formulations, the parameter a represents the radius of the resulting circular geometry ($r = |a|$), meaning the maximum extension or diameter of the shape is equal to 2a.




- The graph of $r = 2a \sin \theta$ represents a circle that passes through the pole and is symmetric with respect to the vertical 90° axis. Its geometric center is located at the polar coordinate point C(a, 90°).
- The graph of $r = 2a \cos \theta$ represents a circle that passes through the pole and is symmetric with respect to the horizontal 0° polar axis. Its geometric center is situated at the polar coordinate point C(a, 0°).

### Cardioids

When a polar equation combines a constant term and a trigonometric term linearly, it alters the distance tracking behavior to produce heart-shaped geometric paths. A **cardioid** is a specific algebraic curve defined by variations of the following standard equations:

| $r = a(1 \pm \sin \theta)$ | $r = a(1 \pm \cos \theta)$ |
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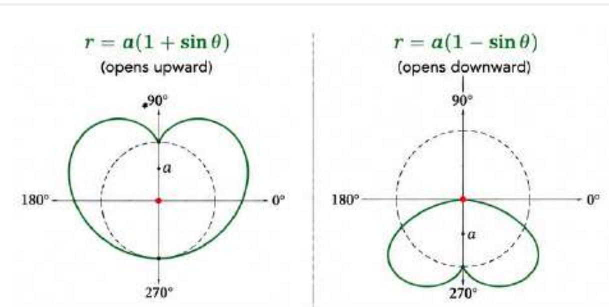
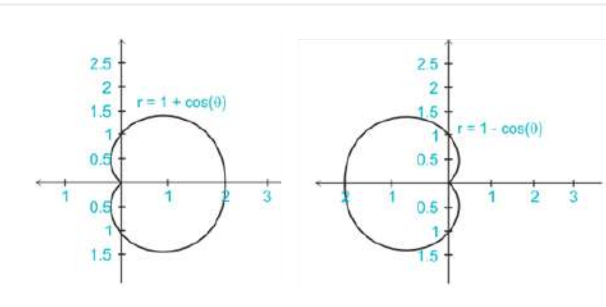
|  |  |

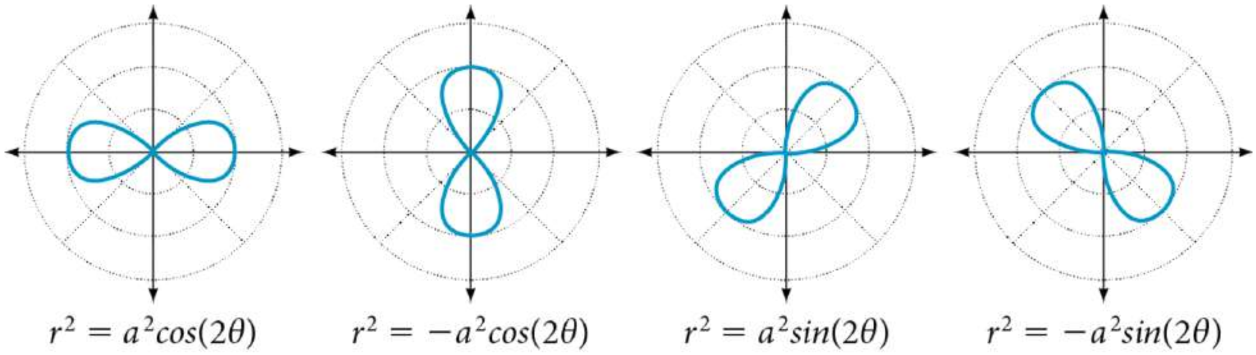




### Lemniscates

When the radius vector is squared within a polar equation and paired with a doubled vectorial angle, the resulting balance creates a classic figure-eight or propeller-shaped path. A **lemniscate** is a loop curve defined mathematically by the equations:

$r^2 = a^2 \sin 2\theta$ and $r^2 = a^2 \cos 2\theta$

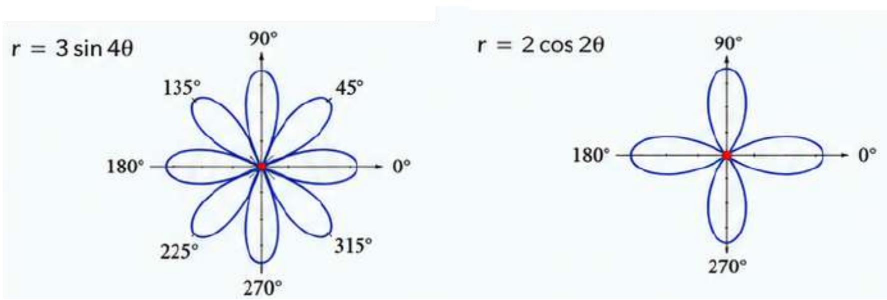



In this equation structure, the parameter $a^2$ represents a scaling coefficient that controls the length of the loops, where the maximum distance from the pole to the tip of either loop is exactly equal to a.

- The graph generated by $r^2 = a^2 \cos 2\theta$ positions its dual symmetrical loops directly along the horizontal reference axis, extending outward along the 0° and 180° terminal lines.
- The graph generated by $r^2 = a^2 \sin 2\theta$ rotates this geometry, positioning its symmetrical loops along the diagonal quadrants at a 45° orientation relative to the primary axes.

**n-Leaved Roses**

Introducing a constant multiplier to the vectorial angle inside a standard trigonometric function produces a collection of symmetric loops radiating outward from the pole. The graph generated by the formulas: $r = a \sin (n\theta)$ and $r = a \cos (n\theta)$




is defined as an **n-leaved rose** (or rose curve), where the scalar value represents the maximum length or reach of each individual petal. The total number of petals present in the complete geometric design depends entirely on the numerical properties of the integer coefficient n:

- **Odd Integer Rule:** If the coefficient n is an **odd integer**, the total number of petals formed matches the integer value exactly, producing an **n-leaved rose**.
- **Even Integer Rule:** If the coefficient n is an **even integer**, the total number of petals formed doubles the integer value, producing a **2n-leaved rose**.
- **The Baseline Case (n = 1):** If the coefficient is exactly equal to unity (n = 1), the mathematical system resolves down to a single-petaled structure. This eliminates the multi-petaled rose configuration entirely, collapsing the path back into a standard **circular** locus that passes through the pole.

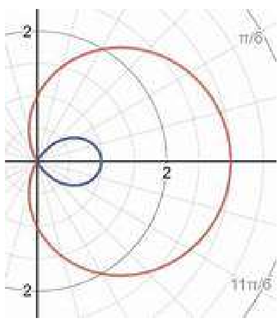
**Limaçons of Pascal**

Expanding upon classical polar loci, the **limaçon of Pascal** is a classic roulette curve formed by tracing a fixed point on a circle as it rolls around the exterior of an identical stationary circle. In the polar coordinate system, this family of curves is defined by the baseline linear equations:

$r = a \pm b \sin\theta$ and $r = a \pm b \cos\theta$

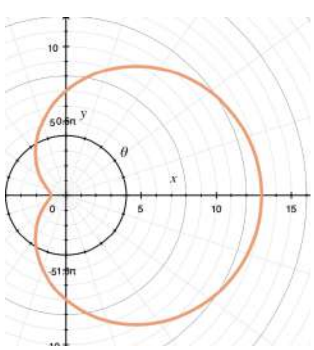
In these structural formulations, a and b represent real constants that dictate the scale and specific geometric morphology of the curve. The trigonometric function governs the orientation of the shape: equations utilizing the cosine function align the curve symmetrically along the horizontal 0°/180° polar axis, while equations utilizing the sine function create a curve that is perfectly symmetrical across the vertical 90°/270° axis. The operational sign ($\pm$) determines the primary direction toward which the bulk of the geometric body extends.

The fundamental geometric profile of a limaçon changes significantly based on the relative magnitudes of its parameters. By analyzing the absolute value ratio |a/b|, the resulting graph falls into three distinct structural categories:




**Limaçon with an Inner Loop (|a| < |b|):** When the absolute value of the constant scalar a is strictly less than the coefficient b, the radius vector r undergoes a change in algebraic sign as θ sweeps through a full rotation. This sign inversion forces the path to cross back through the origin (the pole), creating a distinct, smaller **inner loop** enclosed within the larger outer boundary.

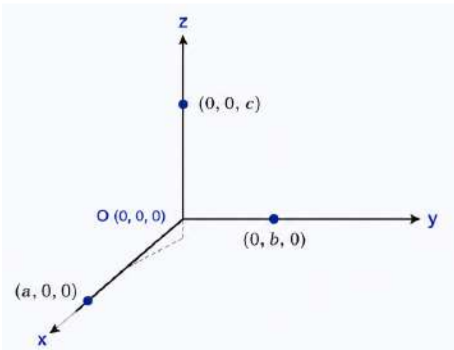
**Cardioid Boundary (|a| = |b|):** When the two parameters are exactly equal in magnitude, the inner loop shrinks down to a single point. The curve develops a sharp, heart-shaped cusp directly at the pole. This establishes a direct geometric link between general limaçons and the **cardioid** configurations discussed in earlier sections.




**Limaçon Without a Loop (|a|> |b|):** When the constant scalar a is strictly greater than the coefficient b, the radius vector r remains entirely positive throughout the entire 360° domain. Because r never drops to zero, the path never touches or passes through the pole. Tracing this equation yields a **loopless limaçon**.

# Three-Dimensional Space Geometry

In space geometry, locating a point in three-dimensional space requires extending the traditional two-dimensional Cartesian plane by introducing a third mutually perpendicular axis.




This framework is constructed using three orthogonal coordinate axes: the x-axis, the y-axis, and the z-axis, which intersect at a single common point known as the origin, O(0,0,0). Any position in this space is uniquely defined by an ordered triplet of real numbers, written as (x, y, z). When a point lies directly on one of these principal axes, its remaining coordinates drop to zero. For example, a point situated along the x-axis at a distance a from the origin is designated as (a, 0, 0), a point along the y-axis at a distance b is (0, b, 0), and a point along the z-axis at a distance c is mapped as (0, 0, c).

**Distance Formula in Three Space**

The straight-line spatial distance d separating two distinct points in a three-dimensional coordinate system, $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, is a direct extension of the two-dimensional distance equation. Derived by applying the Pythagorean theorem across two orthogonal planes, the universal three-space distance formula is expressed as:

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$

**Midpoint Formula**

When finding the absolute geometric center between two terminal spatial positions $P_1$ and $P_2$, the coordinates of the midpoint are calculated by finding the independent arithmetic means of their respective horizontal, vertical, and depth components. The operational equations for the midpoint are written as:

$$x = \frac{x_2+x_1}{2} \quad ; \quad y = \frac{y_2+y_1}{2} \quad ; \quad z = \frac{z_2+z_1}{2}$$

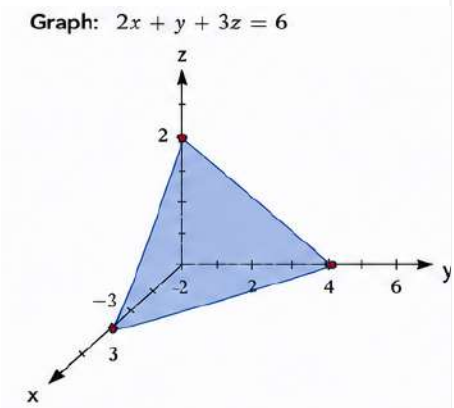
**Distance Between a Point and a Plane**

Calculating the shortest perpendicular distance d from a specific point in space, $P_1(x_1, y_1, z_1)$, to a defined flat plane governed by the general linear equation Ax + By + Cz + D = 0, is a crucial calculation for spatial analysis. The formula computes the absolute value of the plane's expression evaluated at that point, divided by the magnitude of the plane's normal vector:

$$d = \frac{\left|Ax_1 + By_1 + Cz_1 + D\right|}{\sqrt{A^2 + B^2 + C^2}}$$

### A. Planes

In a three-dimensional coordinate system, a first-degree linear equation involving up to three variables defines a flat, infinitely extending surface known as a **plane**. The general algebraic expression representing a plane is written as: $Ax + By + Cz = D$

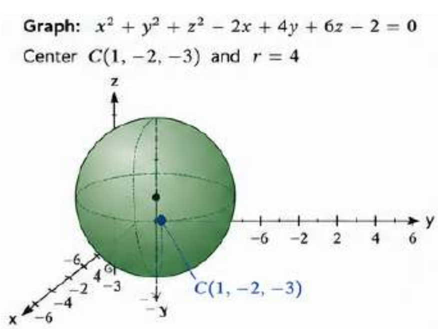



Where A, B, and C serve as components of a vector perpendicular to the surface. To sketch or analyze a plane efficiently during an exam, determine its intercept points where it cuts through the principal axes. For example, the plane defined by $2x + y + 3z = 6$ yields an x-intercept at (3,0,0), a y-intercept at (0,6,0), and a z-intercept at (0,0,2), forming a triangular boundary face when plotted in the first octant.

### B. Spheres

A **sphere** represents the locus of all points in three-dimensional space that maintain a fixed, constant distance (the radius, r) from a designated central point. When a sphere's center is shifted away from the origin to a general spatial position C(h, k, l), its geometry is modeled by the **Standard Equation of a Sphere**: $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$

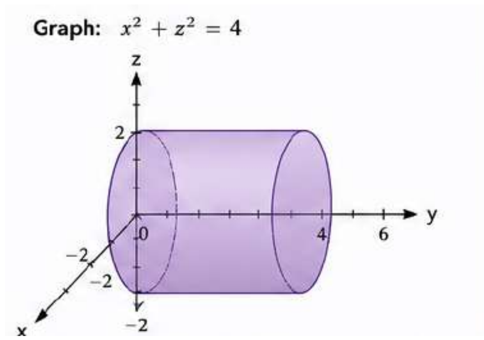
Expanding this standard form algebraically yields the quadratic polynomial known as the **General Equation of a Sphere**: $x^2 + y^2 + z^2 + Dx + Ey + Fz + G = 0$

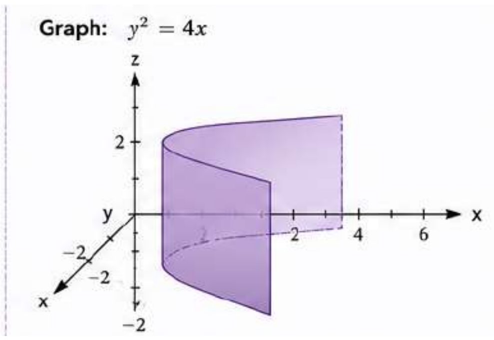



To find the center coordinates and radius from a general equation, you must complete the square for each variable. For instance, given the general equation $x^2 + y^2 + z^2 - 2x + 4y + 6z - 2 = 0$, completing the square transforms the expression into $(x-1)^2 + (y+2)^2 + (z+3)^2 = 16$, revealing a center located at C(1, -2, -3) and a radius of r = 4.

### Cylindrical Surfaces

In three-dimensional space geometry, an equation that omits one of the three principal variables does not merely describe a simple flat curve. Instead, it defines a three-dimensional **cylindrical surface**. This surface is generated by taking a base plane curve (the generating curve) and extending it infinitely along straight paths (rulings) that run parallel to the axis of the missing coordinate variable.

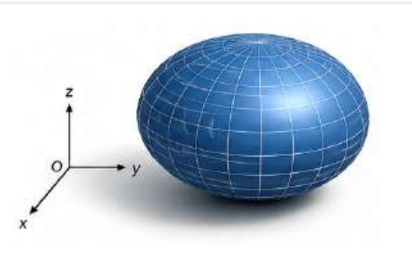



- **Case 1: $x^2 + z^2 = 4$:** Because the variable y is entirely missing from the equation, the circular path drawn in the xz-plane is projected infinitely along the direction of the y-axis. This creates a classic open-ended right circular cylinder centered along the y-axis.
- **Case 2: $y^2 = 4x$:** Because the variable z is missing, the parabolic path drawn in the xy-plane is projected infinitely up and down parallel to the z-axis, generating a parabolic cylindrical gutter.

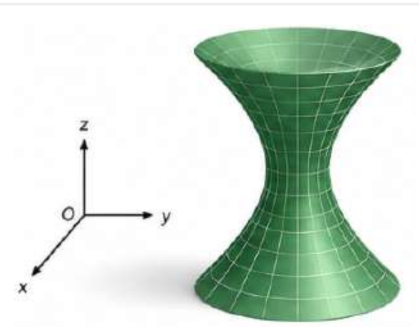
### Quadratic Surfaces

Quadratic surfaces are three-dimensional figures described by second-degree polynomial equations. The specific arrangement of positive and negative signs across the quadratic terms determines the overall shape of the surface:



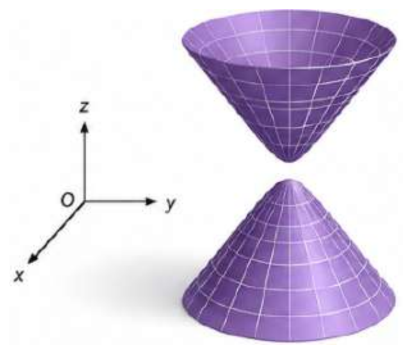

**Ellipsoid:** An **ellipsoid** is a closed surface where all cross-sections parallel to the coordinate planes form ellipses. Its standard equation features three positive quadratic terms set equal to unity:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



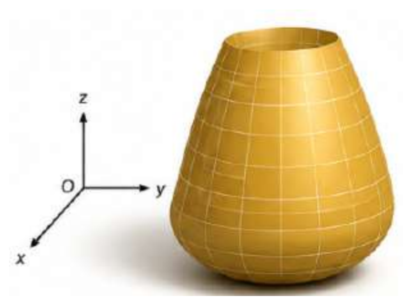

**Hyperboloid of One Sheet:** When one of the three quadratic terms carries a negative sign, the surface becomes a connected, hourglass-like figure known as a **hyperboloid of one sheet**. The axis associated with the negative term dictates the central direction along which the hourglass opens:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$




**Hyperboloid of Two Sheets:** When two of the three quadratic terms carry negative signs, the equation describes a **hyperboloid of two sheets**. This shapes the surface into two separate, mirror-image bowls facing away from each other. The single positive term indicates the axis along which the bowls open:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$




**Elliptic Paraboloid:** An **elliptic paraboloid** is a bowl-shaped surface where sections parallel to the horizontal plane form ellipses, while sections cut along the vertical planes form parabolas. Its standard equation balances two positive quadratic variables against a single linear variable:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$$