DIFFERENTIAL CALCULUS

Differential calculus is a fundamental branch of mathematics concerned with the study of rates of change. Specifically, it analyzes how a change in an independent variable affects a dependent variable, expressing this relationship as the time rate of change of one variable with respect to another. Central to this field is the concept of the derivative, which mathematically measures the instantaneous rate of change or the slope of the tangent line to a curve at any given point.

The formal mathematical definition of the derivative is established through a limit process. Given a continuous function, the derivative is expressed using the difference quotient as the increment interval h approaches zero. This foundational limit formula is written as follows:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In standard engineering notation, the output of this limit is denoted as $\frac{dy}{dx}$ (Leibniz's notation) or simply as y' (Lagrange's prime notation), representing the definitive rate of change of the dependent variable y with respect to the independent variable x.

Differentiation of Algebraic Functions

To analyze physical and geometric systems efficiently without evaluating limits from scratch, differential calculus utilizes specialized rules for the differentiation of algebraic functions. Five core operational rules are presented below:

Derivative of a Constant

The most basic rule states that the **derivative of a constant is zero**. Geometrically, a constant function plots as a flat, horizontal line, which possesses a slope of zero at every point along its domain. Mathematically, if y = c, where c represents any real number or fixed parameters, its rate of change is non-existent:

$$y' = 0$$

The Power Rule

The **Power Rule** serves as the primary mechanism for differentiating variable terms raised to fixed exponents. When a variable or a differentiable function u is raised to a power n, its derivative is found by multiplying the expression by the exponent n and then reducing the original exponent by exactly one unit. In its general form incorporating the Chain Rule, the differential is expressed as:

$$d\left(u^n\right) = n\,u^{n-1}du$$

Illustration: Consider the function y = $2x^{30}$ - $5x^{1/5}$. Applying the power rule to each term individually yields:

$$y' = 2(30)x^{30-1} - 5\left(\frac{1}{5}\right)x^{\frac{1}{5}-1}$$

$$y' = 60x^{29} - x^{-\frac{4}{5}}$$

$$y' = 60x^{29} - \frac{1}{x^{4/5}}$$

Derivative of a Product

When a function is composed of two or more distinct variable expressions multiplied together, the **Derivative of a Product** rule must be employed. The rule states that the derivative of a product of two functions is equal to the **first function times the derivative of the second, plus the second function times the derivative of the first**. This fundamental relationship is formulated as:

$$d(u\,v) = u\,dv + v\,du$$

Illustration: Consider the function y = $(7x^2 - 2)(x^2 + 1)$. Here, let u = $7x^2$ - 2 and v = x^2 + 1. Differentiating each component yields du = 14x and dv = 2x. Substituting these into the product rule gives:

$$y' = \left(7x^2 - 2\right)(2x) + \left(x^2 + 1\right)(14x)$$

$$y' = \left(14x^3 - 4x\right) + \left(14x^3 + 14x\right)$$

$$y' = 28x^3 + 10x$$

Derivative of a Quotient

When an algebraic expression is structured as a fraction where both the numerator and denominator contain variable functions, the **Derivative of a Quotient** rule must be implemented. The verbal rule for this operation dictates that the derivative is equal to the **denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator**. The mathematical formula is structured as follows:

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$$

Illustration: $y = \frac{x-3a}{5x-7a}$, where a is treated as a constant parameter. Let u = x - 3a (implying du = 1) and v = 5x - 7a (implying dv = 5). Substituting these values directly into the quotient formula yields:

$$y' = \frac{(5x-7a)(1)-(x-3a)(5)}{(5x-7a)^2}$$

Simplifying;

$$y' = \frac{5x-7a-5x+15a}{(5x-7a)^2} = \frac{8a}{(5x-7a)^2}$$

The Chain Rule

The **Chain Rule** is a powerful principle used to find the derivative of a composite function, where one function is nested inside another. It allows an engineer to break a complex expression down by differentiating an outer function with respect to an intermediate variable, and then multiplying it by the derivative of that inner intermediate variable with respect to the primary independent variable. This sequential tracking of rates of change is expressed as:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Illustration: Consider a composite system where y = u^3 and the intermediate function is defined as u = x^2 - 3x + 1. The objective is to evaluate the total derivative y' with respect to x. First, compute the separate component derivatives:

$\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 2x - 3$

Applying the Chain Rule, multiply the two resulting derivatives together:

$$y' = \frac{dy}{dx} = \left(3u^2\right)(2x - 3)$$

To finalize the derivative so it is expressed completely in terms of the primary independent variable x, substitute the original definition of u back into the bracketed statement:

$$y' = 3(x^2 - 3x + 1)^2(2x - 3)$$

Implicit Differentiation

In many engineering and mathematical applications, functions are not presented in an explicit format where the dependent variable is isolated entirely on one side of the equation (i.e., y = f(x)). Instead, the relationship between variables is deeply intertwined within an equation, meaning y is not explicitly expressed in terms of x. To determine the rate of change $\frac{dy}{dx}$ for these implicit functions, the technique of **Implicit Differentiation** must be utilized.

This process requires differentiating every single term in the equation with respect to the primary independent variable x. Because y is treated as an implicit function of x, applying the Chain Rule dictates that whenever a term containing y is differentiated, it must be multiplied by the differential operator $\frac{dy}{dx}$ (or y'). Once the differentiation of all terms is complete, algebraic manipulation is used to isolate and solve for $\frac{dy}{dx}$

Illustration: Consider the implicit algebraic expression $2y^2 + 7x - 3y + 5 = 0$, where the goal is to evaluate y'. Differentiating each term sequentially with respect to x yields:

$$\frac{d}{dx}\left(2y^2\right) + \frac{d}{dx}(7x) - \frac{d}{dx}(3y) + \frac{d}{dx}(5) = \frac{d}{dx}(0)$$

$$4y\left(\frac{dy}{dx}\right) + 7 - 3\left(\frac{dy}{dx}\right) + 0 = 0$$

Next, group all terms containing the factor $\frac{dy}{dx}$ on the left-hand side of the equation, and transfer the remaining independent constant terms to the right-hand side:

$$\frac{dy}{dx}(4y - 3) = - 7$$

Factor out the common operator $\frac{dy}{dx}$ from the grouped terms:

$$y' = \frac{dy}{dx} = \frac{-7}{4y-3} = \frac{7}{3-4y}$$

Higher Derivatives

The first derivative $\frac{dy}{dx}$ calculates the instantaneous rate of change of a function at any given point. However, engineers frequently need to monitor how this rate of change itself varies over time or space (such as analyzing acceleration from a velocity function, or monitoring structural beam deflection curves). The derivative of the first derivative, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, is called the **second derivative of y with respect to x**, and is mathematically written as $\frac{d^2y}{dx^2}$ or denoted by y''. This iterative process can continue infinitely to yield third (y'''), fourth (y^4), and higher n^{th} order derivatives.

Illustration: Given the cubic polynomial function $y = 4x^3 - 2x^2 + x - 8$, determine its corresponding second derivative y''. First, apply the standard Power Rule to find the first derivative:

$$y' = 12x^2 - 4x + 1$$

To find the higher-order second derivative, differentiate the expression for y' a second time with respect to x:

$$y'' = \frac{d}{dx}(12x^2 - 4x + 1) = 24x - 4$$

Derivative of Transcendental Functions

Transcendental functions extend beyond basic algebraic expressions, incorporating trigonometric variations, their inverses, and logarithmic tracking systems. Differentiating these functions requires strict adherence to standard transcendental limits combined with the Chain Rule.

Derivative of Trigonometric Functions

Trigonometric derivatives describe periodic physical behavior, such as wave mechanics and alternating current. Assuming u represents a differentiable function of x, the governing formulas for the six primary and reciprocal trigonometric operations are defined as follows:

- **Sine Function:** The derivative of the sine of an angle is the cosine of that angle multiplied by the derivative of the angle itself.

  $$\frac{d}{dx} sin\, u = cos\, u \frac{du}{dx}$$

- **Cosine Function:** Differentiating a cosine function yields a negative sine function, adjusted by the internal derivative.

  $$\frac{d}{dx} cos\, u = - sin\, u \frac{du}{dx}$$

- **Tangent Function:** The instantaneous rate of change of a tangent function is equal to the secant squared of the argument.

  $$\frac{d}{dx} tan\, u = sec^2\, u \frac{du}{dx}$$

- **Cosecant Function:** The derivative of the reciprocal cosecant yields a negative product of the cosecant and cotangent.

  $$\frac{d}{dx} csc\, u = - csc\, u\, cot\, u \frac{du}{dx}$$

- **Secant Function:** Differentiating a secant expression results in the product of the secant and tangent functions.

  $$\frac{d}{dx} sec\, u = sec\, u\, tan\, u \frac{du}{dx}$$

- **Cotangent Function:** The derivative of a cotangent function results in a negative cosecant squared expression.

  $$\frac{d}{dx} cot\, u = - csc^2\, u \frac{du}{dx}$$

Derivative of Inverse Trigonometric Functions

Inverse trigonometric functions (often called arc-functions) map side ratios back to foundational angles. Their derivatives reveal algebraic fractions that do not contain trigonometric terms:

- **Arc-Sine Function:**

$$\frac{d}{dx} arcsin\, u = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

- **Arc-Cosine Function:**

$$\frac{d}{dx} arccos\, u = -\frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$$

- **Arc-Tangent Function:**

$$\frac{d}{dx} arctan\, u = \frac{1}{1+u^2} \frac{du}{dx}$$

- **Arc-Cotangent Function:**

$$\frac{d}{dx} arccot\, u = -\frac{1}{1+u^2} \frac{du}{dx}$$

- **Arc-Secant Function:**

$$\frac{d}{dx} arcsec\, u = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$$

- **Arc-Cosecant Function:**

$$\frac{d}{dx} arccsc\, u = -\frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}$$

Derivative of Logarithmic Functions

Logarithmic differentiation tracking models exponential growth systems and signal decibel conversions. Depending on whether the logarithm uses a general base a or the natural base e, two foundational formulas apply:

- **General Logarithm (Base a):** The derivative of a logarithm with base a is equal to the reciprocal of the internal argument u, multiplied by the derivative of u, and adjusted by the constant scaling factor $\log_a e$.

$$\frac{d}{dx}log_a u = \frac{1}{u}\left(\frac{du}{dx}log_a e\right)$$

- **Natural Logarithm (Base e):** Since ln u represents a logarithm with base e, and ln e = 1, the scaling factor disappears, simplifying the derivative to the reciprocal of the function multiplied by its internal derivative.

$$\frac{d}{dx}ln\, u = \frac{1}{u}\left(\frac{du}{dx}\right)$$

Derivative of Exponential Functions

Exponential functions describe physical processes where the rate of growth or decay is directly proportional to the current quantity, such as population dynamics, radioactive decay, or compound interest equations. In differential calculus, exponential derivatives take two primary structural forms depending on the base of the exponent:

- **General Exponential Base (a^u):** When a constant base a is raised to a variable function u, the derivative reproduces the entire original exponential expression, scaled by the natural logarithm of the base (ln a) and multiplied by the internal derivative of the exponent $\left(\frac{du}{dx}\right)$.

$$\frac{d}{dx}a^u = a^u\frac{du}{dx}\, ln\, a$$

- **Natural Exponential Base (e^u):** The natural exponential base e is a fundamental constant in calculus because the function e^x acts as its own derivative. When generalized using the Chain Rule for a functional power u, the derivative yields the original exponential expression multiplied exclusively by the internal derivative of the exponent $\left(\frac{du}{dx}\right)$.

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$

Derivative of Hyperbolic Functions

Hyperbolic functions arise naturally in engineering applications, such as calculating the tension and shape of a hanging catenary cable (like transmission lines) or modeling structural arch designs. Their derivatives share striking similarities with standard trigonometric derivatives, though they exhibit crucial sign changes:

- **Hyperbolic Sine Function:** The derivative of the hyperbolic sine (sinh u) yields a positive hyperbolic cosine (cosh u).

$$\frac{d}{dx}sinh\, u = cosh\, u\,\frac{du}{dx}$$

- **Hyperbolic Cosine Function:** Unlike standard circular trigonometry where the derivative of cosine is negative, the derivative of the hyperbolic cosine (cosh u) yields a **positive** hyperbolic sine (sinh u).

$$\frac{d}{dx} cosh\, u = sinh\, u \frac{du}{dx}$$

- **Hyperbolic Tangent Function:** Differentiating the hyperbolic tangent (tanh u) yields the hyperbolic secant squared ($sech^2$ u).

$$\frac{d}{dx} tanh\, u = sech^2\, u \frac{du}{dx}$$

- **Hyperbolic Cosecant Function:** The derivative of the hyperbolic cosecant (csch u) yields a negative product of the function and the hyperbolic cotangent (coth u).

$$\frac{d}{dx} csch\, u = - csch\, u\, coth\, u \frac{du}{dx}$$

- **Hyperbolic Secant Function:** Differentiating the hyperbolic secant (sech u) yields a negative product of the function and the hyperbolic tangent (tanh u).

$$\frac{d}{dx} sech\, u = - sech\, u\, tanh\, u \frac{du}{dx}$$

- **Hyperbolic Cotangent Function:** The derivative of the hyperbolic cotangent (coth u) yields a negative hyperbolic cosecant squared ($csch^2$ u).

$$\frac{d}{dx} coth\, u = - csch^2\, u \frac{du}{dx}$$

Variable Power of a Variable (Logarithmic Differentiation)

When an expression consists of a variable base raised to a variable exponent (modeled as y = u^v, where both u and v are functions of x), standard power rules and exponential rules cannot be applied individually. Instead, a specialized composite formula derived via logarithmic differentiation must be deployed:

$$y = u^v \qquad \Rightarrow \qquad \frac{dy}{dx} = y\left[\frac{v}{u}\left(\frac{du}{dx}\right) + ln\, u\left(\frac{dv}{dx}\right)\right]$$

Substituting u^v back into the formula yields the standard expression:

$$\frac{dy}{dx} = u^v \left[\frac{v}{u}\left(\frac{du}{dx}\right) + ln\, u\left(\frac{dv}{dx}\right)\right]$$

Illustration: Consider the function $y = x^{x^2}$. Let the base function be u = x (implying $\frac{du}{dx}$ = 1) and the exponent function be v = x^2 (implying $\frac{dv}{dx}$ = 2x). Substituting these terms into the variable power formula yields:

$$\frac{dy}{dx} = x^{x^2}\left[\frac{x^2}{x}(1) + \ln x(2x)\right]$$

Simplify the internal components within the brackets by reducing the algebraic fraction $\frac{x^2}{x}$ to x:

$$\frac{dy}{dx} = x^{x^2}\ [x + 2x \ln x]$$

To achieve the simplified presentation highlighted in the review card, factor out the common variable x from both terms inside the bracket:

$$\frac{dy}{dx} = x^{x^2}\ \cdot x\,[1 + 2 \ln x]$$

Applying basic laws of exponents ($x^{x^2} \cdot x^1 = x^{x^2+1}$), combine the outside factors to reach the final derivative expression:

$$y' = x^{x^2+1}\ [1 + 2 \ln x]$$

Maxima and Minima in a Graph (Curve Sketching)

Differential calculus provides the foundation for optimization analysis by enabling engineers to mathematically pinpoint peak efficiencies (maxima) or cost boundaries (minima) along a continuous system curve.

Stationary Turning Points

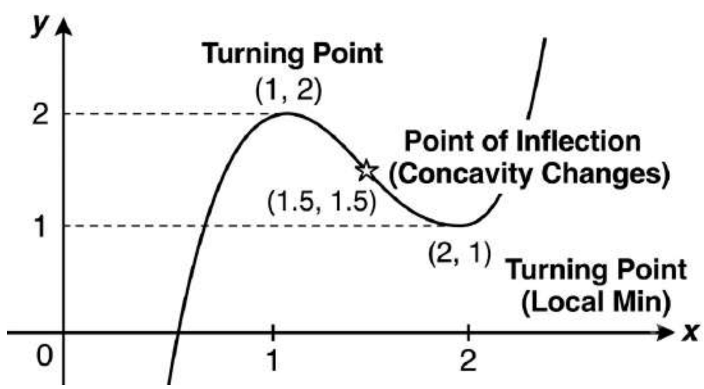
Maximum and minimum points are structurally known as **turning points**. Geometrically, these points represent locations along a curve where the trajectory shifts from an upward slope to a downward slope (a local maximum) or from a downward slope to an upward slope (a local minimum). Because the tangent line at the exact peak or valley of a continuous curve is perfectly horizontal, the rate of change or slope is non-existent. Therefore, at the maximum and minimum points in the graph, the slope is defined as: $\frac{dy}{dx} = 0$

Points of Inflection

A **point of inflection** is distinct from a turning point. It is defined as the point on the curve where the direction of the curve changes (the graph shifts its concavity from concave upward to concave downward, or vice versa). An inflection point does not represent a peak or a valley; therefore, it is explicitly not the maximum and minimum point. At the point of inflection, the second derivative equals zero, while the first derivative remains non-zero:

$$\frac{d^2y}{dx^2} = 0 \qquad \text{Note: } \frac{dy}{dx} \neq 0$$

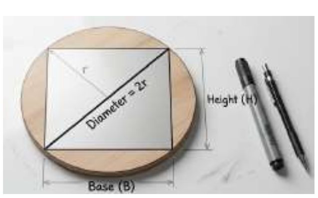
Illustration: Evaluate the provided function: y = $2x^3 - 9x^2 + 12x - 3$, calculating its first derivative yields y' = $6x^2 - 18x + 12$ Setting y' = 0 isolates the turning coordinates. From the structural graph provided, the local maximum point is explicitly located at the coordinates **(1, 2)**, and the corresponding local minimum point rests at the coordinates **(2, 1)**.



Common Maxima and Minima Geometric Relationships

In engineering board examinations, optimization problems involving geometric figures occur frequently. While these problems can be solved from scratch by setting up a primary optimization equation, differentiating it, and setting the first derivative to zero, certain classic geometric configurations yield standard, recurring mathematical shortcuts. Memorizing these common variable relationships for maximum or minimum values provides a critical speed advantage during timed examinations.

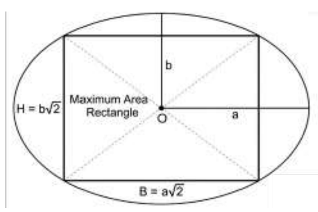
1. Maximum Dimension of a Rectangle Cut from a Circle



When maximizing the area of a rectangle that can be carved out of a circle of radius r, the optimal shape is always a perfect square. This uniform behavior means that the optimal base (B) and optimal height (H) are mathematically identical. By applying the Pythagorean theorem across the diagonal of the rectangle (which forms the circle's diameter, 2r), the optimization shortcut simplifies to:

$$B = H = r\sqrt{2}$$

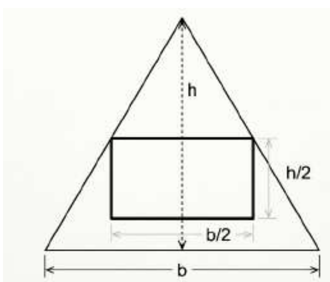
2. Maximum Dimension of a Rectangle Inscribed in an Ellipse



For a rectangle inscribed inside an ellipse defined by the semi-major axis a and semi-minor axis b, the maximum area occurs when the rectangle's dimensions scale proportionally to the axes of the host ellipse. The optimal base (B) and optimal height (H) of this maximum-area rectangle are governed by the following equations:

$$B = a\sqrt{2} \quad \text{and} \quad H = b\sqrt{2}$$

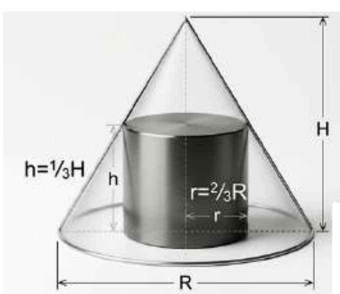
3. Maximum Rectangle Cut from a Triangle



When inscribing a rectangle of base x and height y within a parent triangle that possesses a base b and altitude h, the maximum possible area of the rectangle is realized when its dimensions are exactly half of the corresponding dimensions of the surrounding triangle. This clean geometric bisection yields the following simple formulas:

$$x = \frac{b}{2} \quad \text{and} \quad y = \frac{h}{2}$$

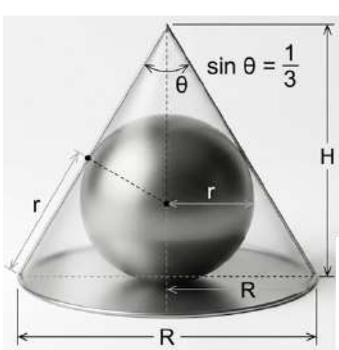
4. Maximum Dimension of a Cylinder Inscribed in a Cone



When a right circular cylinder of radius r and height h is inscribed inside a right circular cone of overall height H and base radius R, maximizing the cylinder's volume relies on a fixed ratio derived from similar triangles. The maximum volume is achieved when the height of the cylinder is exactly one-third of the cone's height, and its radius constitutes exactly two-thirds of the cone's base radius:

$$h = \frac{1}{3}H \quad \text{and} \quad r = \frac{2}{3}R$$

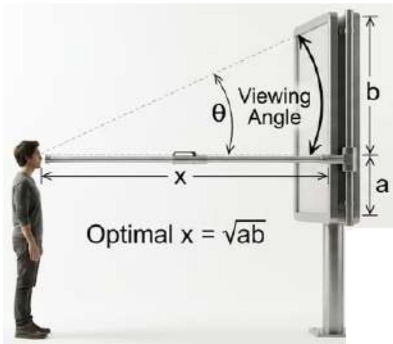
5. Smallest Cone Circumscribing a Sphere



In minimization problems, finding the smallest volume or surface area of a right circular cone that can circumscribe a rigid sphere of radius r is a classic problem. Let θ represent the semi-vertical angle (the angle between the vertical axis of symmetry and the slanted side of the cone). The optimized minimum-volume cone is locked into a fixed geometric ratio where the sine of this critical angle is always equal to exactly one-third:

$$sin\,\theta = \frac{1}{3}$$

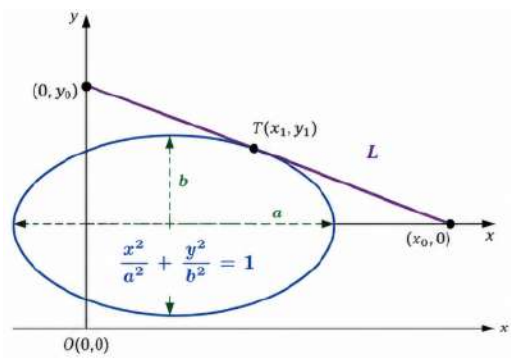
6. Maximum Viewing Angle (θ)



A classic calculus problem involves optimizing the position of an observer standing on the ground to achieve the maximum viewing angle θ) of a vertical object (such as a painting, billboard, or statue) whose lower edge is at a height a above eye level and whose upper edge is at a height b above eye level. The optimal horizontal distance x from the observer's eye to the vertical plane of the object maximizes the angle θ when it is equal to the geometric mean of the two heights:

$$x = \sqrt{ab}$$

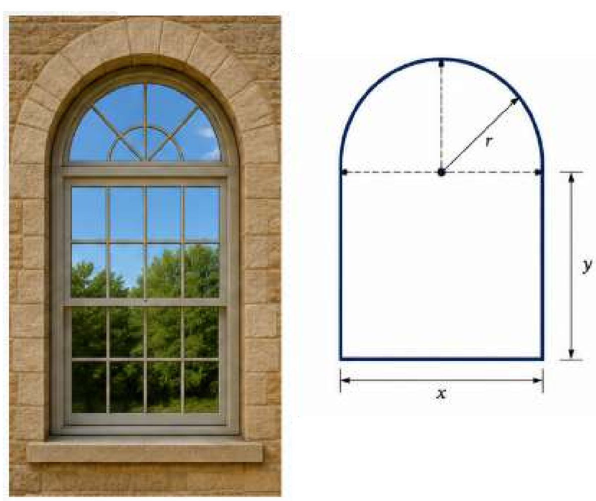
7. Minimum Length (L) of a Line Tangent to an Ellipse



When analyzing a straight line segment that is tangent to an ellipse in the first quadrant and bounded by the horizontal and vertical coordinate axes, calculating its minimum possible length L is a classic optimization problem. If the ellipse is defined by a semi-major axis a and a semi-minor axis b, the minimum length L cannot be a simple sum of the two axes. Instead, it is governed by the structural relationship:

$$L^{2/3} = a^{2/3} + b^{2/3}$$

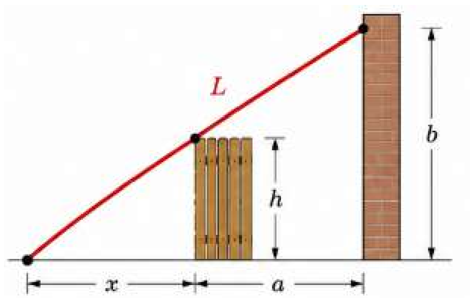
8. Maximum Area of a Norman Window of Given Perimeter



A Norman window consists of a rectangular lower region surmounted by a semi-circular top archway. In architectural optimization problems, the goal is often to find the dimensions that maximize the area to admit the most light for a fixed, given perimeter. Let x represent the total width of the window base, y represent the height of the rectangular side, and r represent the radius of the semi-circle. Because the base width spans the full diameter of the circle, the radius is inherently locked as $r = \frac{x}{2}$. To maximize the window's total area under a constrained perimeter, optimization calculus proves that the height of the rectangular portion must equal the semi-circular radius:

$$y = r = \frac{x}{2}$$

9. Minimum Length of a Ladder Leaning Against a Wall



A traditional optimization scenario involves finding the minimum length L of a rigid ladder that can reach a tall vertical wall while clearing a vertical fence or obstruction of height h situated at a horizontal distance a from the wall. Let the horizontal distance from the base of the wall to the fence be denoted as a constant a, and the height of the fence be a constant h. The minimum length L of the ladder required to span from the ground, over the top of the fence, to the surface of the wall forms a structural envelope relationship known mathematically as:

$$L^{2/3} = h^{2/3} + b^{2/3}$$

Time Rates

In physical and engineering systems, variables rarely remain static; instead, they change continuously over time. **Time Rates** problems utilize differential calculus to determine the instantaneous rate at which a dependent variable changes with respect to time (t). By applying the Chain Rule, multiple related rates within a geometric or physical system can be linked together. Four foundational time rates frequently encountered in engineering dynamics and fluid mechanics are defined below:

Velocity

Velocity (v) measures the instantaneous rate of change of linear displacement or distance with respect to time. When an object travels along a path, its position at any given moment can be modeled by a position function s. Differentiating this position function with respect to time yields the velocity:

$$v = \frac{ds}{dt}$$

Acceleration

Acceleration (a) tracks the instantaneous rate of change of an object's velocity with respect to time. It indicates whether an engineering system is speeding up, slowing down, or changing direction. Because it evaluates how velocity changes over time, it represents the first derivative of velocity, or conversely, the second derivative of the position function (s):

$$a = \frac{dv}{dt}$$

Discharge (Volumetric Flow Rate)

In fluid mechanics and hydraulic engineering, discharge (Q) measures the volumetric flow rate of a fluid system. It describes the instantaneous rate at which the volume (V) of a fluid passes through a given cross-sectional area or accumulates inside a storage vessel over time. Letting V represent the volume at any given time, the relationship is formulated as:

$$Q = \frac{dV}{dt}$$

Angular Speed

Angular speed (ω) tracks rotational dynamics, measuring the instantaneous rate at which an object rotates or revolves through a central angle. Instead of measuring linear distance, it calculates the change in the angular displacement (θ) over time. To maintain mathematical consistency across physics and engineering calculations, the angular position θ must be evaluated at any time in radians (rad), yielding the rate:

$$\omega = \frac{d\theta}{dt}$$