INTEGRAL CALCULUS

Integral calculus is a branch of mathematics used to determine volumes bounded by surfaces, areas bounded by curves, equations of curves from their rates of change, and analytical solutions to differential equations. By shifting perspective from finding an instantaneous rate of change to accumulating continuous quantities, integration allows engineers to compute structural properties like centroids, moments of inertia, and total fluid hydrostatic force.

The Power Rule for Integration

The most heavily utilized rule in algebraic integration is the Power Rule. It reverses the operation of the differential power rule by adding an integer or rational value of one to the exponent and dividing the expression by this newly computed exponent. Because an indefinite integral lacks specified boundaries, it represents an entire family of curves shifted vertically; thus, a constant of integration (C) must always be appended.

The general mathematical formula is defined as:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad ; \qquad n \neq -1$$

The formula explicitly states a restriction where $n \neq -1$. If n = -1, the denominator evaluates to zero $\left(\frac{x^0}{0}\right)$, creating an undefined mathematical operation. When an exponent is exactly -1, the integration shifts from the Power Rule to a logarithmic form.

Illustration:

Consider the following integral problem:

$$\text{Integrate } \int \frac{2x^2 + 1}{\sqrt[3]{\left(2x^3 + 3x + 1\right)^2}} dx$$

To solve this systematically, analyze the expression to find an inner function whose derivative looks structurally identical to the remaining terms. Let the inner polynomial expression of the denominator be defined as:

$$u = 2x^3 + 3x + 1$$

Differentiating u with respect to x yields its corresponding differential component:

$$du = \left(6x^2 + 3\right)dx$$

Factoring out a common scalar multiplier of 3 reveals the direct relationship to our numerator:

$$du = 3\left(2x^2 + 3\right)dx$$

$$\frac{du}{3} = \left(2x^2 + 3\right)dx$$

Now, substitute u and $\frac{du}{3}$ directly back into the original integration problem. Convert the radical expression in the denominator into a negative fractional exponent $\left(\frac{1}{u^{2/3}} = u^{-2/3}\right)$ and pull out the constant factor of $\frac{1}{3}$:

$$\int \frac{1}{u^{2/3}} \cdot \frac{du}{3} = \frac{1}{3}\int \frac{1}{u^{2/3}} du = \frac{1}{3}\int u^{-2/3} du$$

Apply the corrected Fundamental Power Rule:

$$u^{1/3} + C$$

Finally, substitute the original expression for u back into the solution to achieve the definitive final answer:

$$\left(2x^3 + 3x + 1\right)^{1/3} + C$$

Integration by Parts

When an integrand consists of a product of transcendental and algebraic functions that cannot be resolved using simple u-substitution, engineers turn to **Integration by Parts**. This technique is derived directly from reversing the Product Rule of differential calculus. It separates the integrand into two distinct sections: a component u that is easily differentiable, and a component dv that is readily integrable.

The operational formula is written as: $\int u\, dv = uv - \int v\, du$

Illustration: Integrate: $\int x^2 e^x\, dx$

Using the LIATE standard priority framework (Logarithmic, Inverse Trig, Algebraic, Trigonometric, Exponential), select the algebraic term to be differentiated and the exponential term to be integrated:

Let: $u = x^2 \quad \Rightarrow \quad du = 2x\, dx$

Let: $dv = e^x dx \quad \Rightarrow \quad v = e^x$

Substitute these choices into the primary integration by parts formula:

$$\int x^2 e^x\,dx = x^2 e^x - \int e^x (2x dx)$$

$$= x^2 e^x - 2\int x e^x dx$$

Notice that the remaining integral, $\int x e^x dx$, still contains a product of an algebraic variable and an exponential function. Therefore, a second, nested application of integration by parts must be carried out:

Let: $u = x \quad \Rightarrow \quad du = dx$

Let: $dv = e^x dx \quad \Rightarrow \quad v = e^x$

Substitute this inner iteration carefully, ensuring the leading negative sign and scalar multiplier of 2 are distributed over the entire secondary expression:

$$= x^2 e^x - 2\left(x e^x - \int e^x dx\right)$$

Evaluate the remaining fundamental integral ($\int e^x dx = e^x$) and distribute the -2 through the parentheses. Remember to append the final integration constant C:

$$= x^2 e^x - 2x e^x + 2e^x + C$$

Integration of Logarithmic and Exponential Functions

This category comprises integrations that result in transcendental logarithmic forms or manipulate continuous base growth exponents. They represent standard formulas that must be thoroughly memorized for rapid recall during examinations:

- **Natural Exponential Function:** The natural exponential function e^u serves as its own basic integral form under differential conditions. $\int e^u\,du = e^u + C$
- **General Exponential Function:** When integrating a constant base a raised to a variable exponent u, the resulting expression reproduces the original exponential expression divided by the natural log of the base. $\int a^u\,du = \frac{a^u}{\ln a} + C$

- **The Reciprocal Power Rule Alternative:** When an algebraic variable resides in the denominator with a power of exactly one (u^{-1}), its integration evaluates directly as a natural logarithm. $\int \frac{1}{u}\,du \;= ln\,|u| \,+\, C$

- **Integral of the Natural Logarithm:** Integrating the natural logarithm function itself requires a standalone integration by parts derivation, establishing a distinct structural pattern. $\int ln\,u\,du \;= u\,ln\,u \;-\; u \,+\, C$

Integration of Trigonometric Functions

Trigonometric integrations are essential for resolving wave equations, harmonic physical motions, and rotational mechanical components. These six fundamental trigonometric integration expressions form the foundation of periodic calculus operations:

$$\int cos\,x\,dx \,=\, sin\,x \,+\, C$$

$$\int sin\,x\,dx \,=-\, cosx \,+\, C$$

$$\int sec^2 x\,dx \,=\, tan\,x \,+\, C$$

$$\int csc\,x\,cot\,x\,dx \,=-\, cscx \,+\, C$$

$$\int sec\,x\,tan\,x\,dx \,=\, sec\,x \,+\, C$$

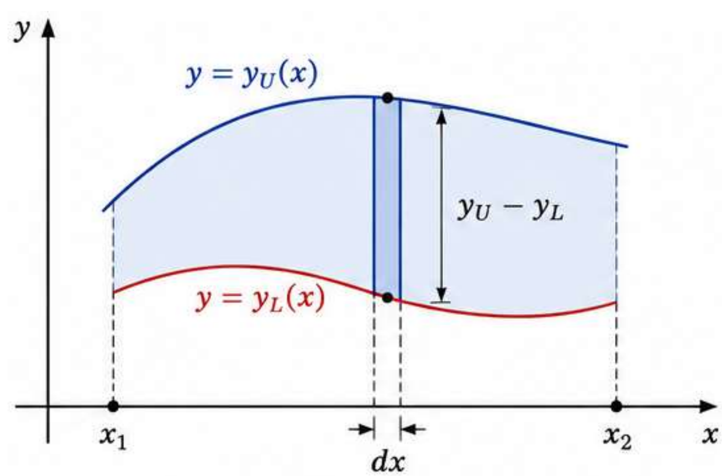
$$\int csc^2 x\,dx \,=-\, cot\,x \,+\, C$$

Definite Integrals: Area Bounded by Curves

A primary geometric application of definite integration in engineering board examinations is the determination of planar areas bounded by intersecting curves. Conceptually, finding the area between two functions involves accumulating an infinite number of infinitesimally thin geometric rectangles (or differential elements of area, dA) across a specific interval. Depending on the geometry of the bounding functions, engineers can establish these differential strips either vertically or horizontally. Choosing the correct orientation is critical for simplifying the algebraic setup and avoiding multi-part integration intervals.

Vertical Differential Strips

When a region is bounded above by an upper curve y_U and below by a lower curve y_L over a continuous horizontal interval from $x = x_1$ to $x = x_2$, a vertical differential strip is selected.



The width of this vertical strip is an infinitesimal change in the horizontal direction, represented by the differential dx. The height of the strip is determined by subtracting the lower y-coordinate from the upper y-coordinate, which yields the expression $\left(y_U - y_L\right)$.

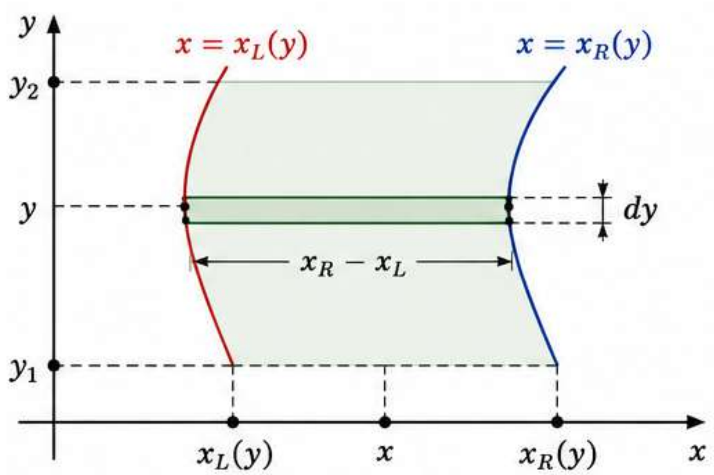
Multiplying the height by the width gives the differential element of area, dA = $\left(y_U - y_L\right)dx$. To find the total area A of the bounded region, this differential element is integrated across the full span of the independent variable x:

$$A = \int_{x_1}^{x_2} \left(y_U - y_L\right) dx$$

In practice, before executing this integration, both boundary functions y_U and y_L must be expressed explicitly in terms of x (i.e., y = f(x)). The integration limits x_1 and x_2 correspond to the x-coordinates of the intersection points of the two curves.

Horizontal Differential Strips

When a geometric region is more cleanly bounded along the lateral sides, it is advantageous to use a horizontal differential strip. The thickness of a horizontal strip represents an infinitesimal change in the vertical direction, denoted by the differential dy.



The length of this strip is calculated by taking the rightmost boundary curve (x_R) and subtracting the leftmost boundary curve (x_L). Multiplying this horizontal length by the vertical thickness yields the differential element of area, dA =$\left(x_R - x_L\right)dy$. Integrating this differential element with respect to the independent variable y from the lowest vertical intersection point y_1 to the highest vertical intersection point y_2 yields the total area formula:

$$A = \int_{y_1}^{y_2} \left(x_R - x_L\right)dy$$

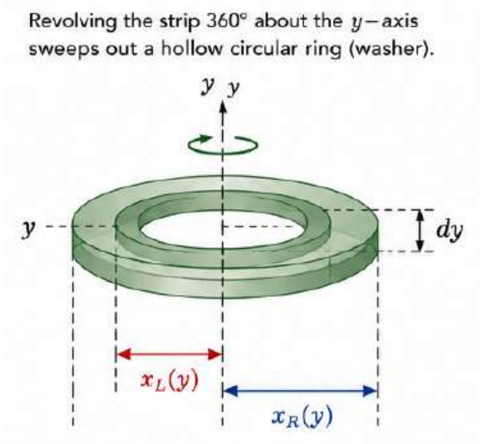
For this method, both boundary functions must be rewritten explicitly in terms of y (i.e., x = g(y)) prior to integration. Choosing this horizontal approach is highly recommended when the bounding equations contain functions that are difficult to isolate for y, or when a vertical strip setup would force the user to split the region into separate integrals.

Volume of Solids of Revolution by Integration

These geometric solids are generated by revolving a defined planar area bounded by curves around a fixed straight line known as the axis of rotation. To evaluate these volumes, engineers break down the solid into differential volume elements (dV) using two primary geometric methods: the Circular Disk (or Washer) Method and the Hollow Cylindrical Shell Method. The optimal selection between these methods depends on the orientation of the chosen differential strip relative to the axis of revolution.

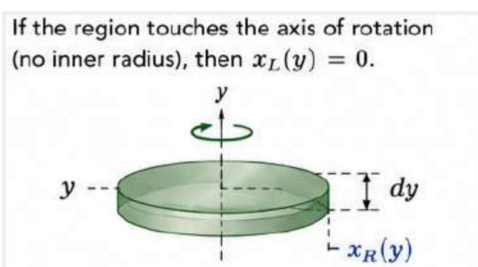
The Circular Disk and Washer Method

When the differential element chosen is perpendicular to the axis of rotation, revolving it sweeps out a circular ring or "washer" containing a distinct hollow core. The volume of this differential washer element is calculated by taking the area of the larger outer circle and subtracting the area of the smaller inner circle, then multiplying by the infinitesimal thickness of the strip.



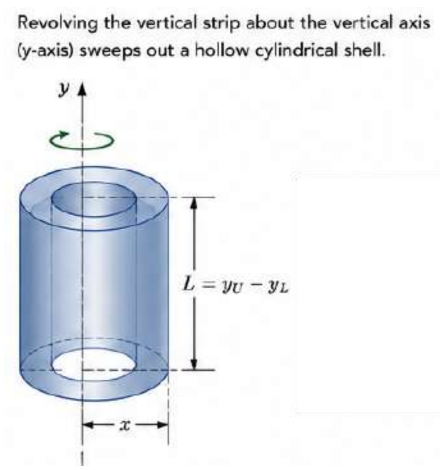
When a region is bounded on the right by a curve x_R and on the left by a curve x_L, and it is rotated around a vertical axis (such as the y-axis), a horizontal differential strip of thickness dy is used. As this strip sweeps 360° around the vertical axis, the outer radius of rotation becomes x_R and the inner radius becomes x_L. The differential volume of this hollowed circular ring is given $dV = \pi\left(x_R^2 - x_L^2\right)$. Summing these differential washers together from the lowest vertical limit y_1 to the highest vertical limit y_2 yields the definite integral:

$$V = \pi\int\left(x_R^2 - x_L^2\right)dy$$



Before integrating, the functions must be expressed explicitly in terms of y. If the region perfectly touches the axis of rotation without a gap, the inner radius x_L becomes zero, reducing this formula to the classic solid circular disk method $V = \pi\int x^2 dy$.

The Hollow Cylindrical Shell Method



When the selected differential strip runs parallel to the axis of rotation, revolving it around that axis generates an incredibly thin, hollow cylindrical shell. The differential volume (dV) of this shell is conceptually determined by unrolling it into a flat rectangular sheet. The volume of this unrolled sheet equals its circumference $(2\pi \cdot radius)$ multiplied by its height (L), multiplied by its thin wall thickness. For a region bounded above by a curve y_U and below by a curve y_L over a horizontal span from $x = x_1$ to $x = x_2$, a vertical differential strip of width dx is established.

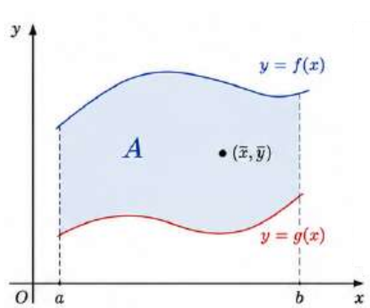
When rotated around a vertical axis (like the y-axis), the distance from the axis to the parallel strip serves as the radius of rotation, which is simply x. The vertical height L of this strip is determined by the functional difference$\left(y_U - y_L\right)$. Multiplying these elements yields the total accumulated volume formula:

$$V = 2\pi \int x\left(y_U - y_L\right)dx$$

This shell method is highly advantageous when the bounding functions are complex to solve inversely, allowing engineers to integrate with respect to x even when revolving around a vertical axis.

Centroid of a Plane Area

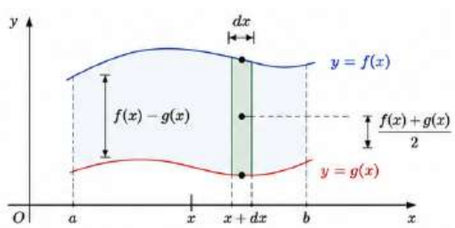
The centroid represents the geometric center of a two-dimensional planar area. In physical mechanics, if the area represents a plate of uniform thickness and density, the centroid coincides exactly with its center of gravity. Board exam problems often require locating these coordinates, denoted as $(\bar{x}, \bar{y})$.



The location of the centroid is determined by using the principle of moments, where the total area A multiplied by its respective centroidal distance equals the first moment of the area about the coordinate axes:

$A\bar{x} = m_y$ and $A\bar{y} = m_x$

First Moment of Area About the x-Axis (m_x)

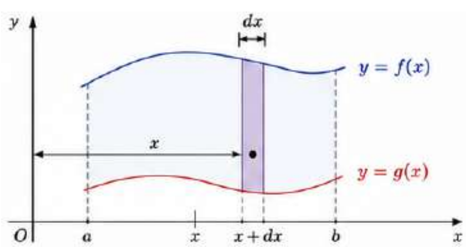


The moment of a planar area with respect to the horizontal x-axis is obtained by accumulating the moments of vertical differential strips. For a region bounded between an upper curve f(x) and a lower curve g(x) from x = a to x = b, the centroid of an individual vertical strip rests exactly halfway between the top and bottom boundaries, at a height of $\frac{f(x)+g(x)}{2}$. The area of that strip is $[f(x) + g(x)]$. Multiplying the strip's area by its centroidal height yields the structural formula for m_x:

$$m_x = \frac{1}{2}\int_a^b \left[f(x)^2 - g(x)^2\right]dx$$

(Note: If the area is bounded below by the x-axis itself, then g(x) = 0, simplifying the integrand directly to the card's format of $\left[f(x)^2\right]$

First Moment of Area About the y-Axis (m_y)



The moment of a planar area with respect to the vertical y-axis measures horizontal leverage. The distance from the y-axis to the center of a vertical differential strip is simply the coordinate variable x. Multiplying this horizontal distance by the differential element of area yields the foundational integration formula for m_y:

$$m_y = \int_a^b x[f(x) - g(x)]dx$$

Once the first moments of area (m_x, m_y) and the total total area A are calculated via standard integration, the exact coordinates of the geometric centroid are determined by division:

$\bar{x} = \frac{m_y}{A}$ and $\bar{y} = \frac{m_x}{A}$

DIFFERENTIAL EQUATIONS

Differential equations represent a foundational branch of mathematics that deals with equations involving an unknown function and one or more of its derivatives. In engineering and physical sciences, dynamic processes are rarely described by static values; instead, they are governed by rates of change. Differential equations serve as the definitive mathematical tool to model these natural phenomena—such as the exponential growth of biological populations, radioactive decay, and fluid cooling rates—as well as highly complex engineering systems, including structural vibrations, electrical circuits, and damped or forced mechanical oscillations. Solving a differential equation means discovering the underlying mathematical function that satisfies these changing rates.

To approach any DE problem systematically, you must first classify the equation. The classification dictates the exact analytical strategy needed to find a solution.

Ordinary vs. Partial Differential Equations

- **Ordinary Differential Equations (ODEs):** An equation containing exclusively ordinary derivatives, where the unknown dependent function depends on a single independent variable (e.g., tracking a single point-mass displacement y over time t). All equations in this review module are ODEs.
- **Partial Differential Equations (PDEs):** An equation containing partial derivatives, where the unknown function is governed by two or more independent variables simultaneously (e.g., heat conduction through a solid body as a function of both position x and time t, represented as $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$).

Order and Degree

- **Order:** The order of a differential equation is the order of the highest derivative present in the equation. A first-order equation contains only y', while a second-order equation contains y''.
- **Degree:** The degree of a differential equation is the algebraic exponent to which the highest-order derivative is raised, provided the equation has been written in a polynomial form with respect to its derivatives.

Example:

For the equation $(y''')^4 + (y'')^5 - (y')^2 = 0$, the highest derivative is the third derivative (y'''). The exponent belonging specifically to that third derivative is 4. Therefore, this is a **third-order, fourth-degree** differential equation.

General vs. Particular Solutions

- **General Solution:** The complete mathematical representation of all possible functions that satisfy the differential equation. Because integration is required to eliminate derivatives, a general solution inherently **contains arbitrary constants** (e.g., C, C_1, C_2). The number of independent arbitrary constants matches the order of the ODE exactly.
- **Particular Solution:** A unique, single functional expression extracted from the general solution. It is derived by evaluating the arbitrary constants using specific, real-world constraints known as **initial conditions** or **boundary conditions** (e.g., specifying that at time t = 0, the initial temperature T = T_0). A particular solution **contains no arbitrary constants**.

First-Order Solution Methods

The standard differential form of a first-order ODE is expressed as:

$$M(x,\ y)dx\ +\ N(x,y)dy\ =\ 0$$

Four specific types of first-order equations are the following:

1. Separation of Variables: An equation is separable if the coefficient functions M and N can be factored into products of single-variable functions, allowing you to group all x terms exclusively with dx, and all y terms exclusively with dy.

Illustration: Solve the initial value problem: $2xy\frac{dy}{dx} = 1 + y^2$ when x=2 and y=3

Step 1: Separate the variables. Multiplied out into differential form, the equation is $2xy\ dy = (1 + y^2)dx$. Rearrange it by dividing both sides by $x(1 + y^2)$ to isolate the matching variables on their respective sides:

$$\frac{2y}{1+y^2}dy = \frac{dx}{x}$$

Step 2: Integrate both sides simultaneously. The left side is integrated using a basic u-substitution where $u = 1 + y^2$ and $du = 2ydy$, which yields a natural logarithm. The right side integrates directly to $ln\ x$. For seamless algebraic manipulation, write the integration constant as $ln\ C$:

$$\int\frac{2y}{1+y^2}dy = \int\frac{dx}{x}$$

$$ln\left(1 + y^2\right) = lnx + ln\ C$$

Step 3: **Simplify via logarithmic properties.** Apply the logarithmic product rule ($ln\, a + lnb = ln(ab)$) to group the right side:

$$ln\left(1 + y^2\right) = ln\, Cx$$

Exponentiate both sides ($e^{ln(...)}$) to clear the logarithms and reveal the general solution:

$$1 + y^2 = Cx$$

$$C = \frac{1+y^2}{x}$$

Step 4: **Apply initial conditions to find the particular solution.** Substitute x=2 and y=3 into your general solution equation to solve for the specific constant C:

$$C = \frac{1+(3)^2}{4} = 5$$

Substitute C = 5 back into the general equation to establish the final, definitive particular solution:

$$5x = 1+y^2$$

2. Exact Differential Equations. An equation is an exact differential equation if the left-hand side matches the total differential of an underlying multivariable function. By definition, the equation is exact if and only if the mixed partial derivatives are identical: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Determine if the following equation is exact: $(3x^2y - 4xy^3 + 6x),dx + (x^3 - 6x^2y^2 - 1)dy = 0$

Step 1: **Identify M and N.**

$$M = 3x^2y - 4xy^3 + 6x$$

$$N = x^3 - 6x^2y^2 - 1$$

Step 2: **Evaluate cross-partial derivatives.** Differentiate M with respect to y (treating x as a constant), and differentiate N with respect to x (treating y as a constant):

$$\frac{\partial M}{\partial y} = 3x^3(1) - 4x\left(3y^2\right) + 0 = 3x^2 - 12xy^2$$

$$\frac{\partial N}{\partial x} = 3x^2 - 6(2x)y^2 - 0 = 3x^2 - 12xy^2$$

Conclusion: Because $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the equation is **explicitly exact**.

3. Linear Differential Equations of Order One A first-order equation is linear if the dependent variable (y) and its derivative (y') appear strictly to the first power, are not multiplied together, and do not reside inside transcendental functions. The standard form is:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

To solve a linear equation, you must compute an **Integrating Factor (δ)**, which is defined as:

$$\delta = e^{\int P(x)dx}$$

Multiplying the standard equation by δ condenses the left side into a perfect product rule derivative, resulting in the final integrable general solution formula:

$$y\delta = \int \delta Q(x)dx + C$$

Solve: $(x^5 + 3y)dx - xdy = 0$

Step 1: Convert to standard linear form. Divide the entire equation by dx to check its profile, then rearrange terms to isolate the derivative at the front:

$$x^5 + 3y - x\frac{dy}{dx} = 0$$

$$-x\frac{dy}{dx} + 3y = -x^5$$

Divide every single term by -x so that the leading coefficient of the derivative equals exactly 1:

$$\frac{dy}{dx} + \frac{3}{x}y = x^4$$

Step 2: Identify P(x) and Q(x).

$P(x) = -\frac{3}{x}$ and $Q(x) = x^4$

Step 3: **Compute the Integrating Factor (δ).**

$$\delta = e^{\int -\frac{3}{x}dx} = e^{-3\ln x} = e^{\ln\left(x^{-3}\right)} = x^{-3}$$

Step 4: **Integrate using the general linear solution formula.** Substitute δ and Q(x) into the framework:

$$yx^{-3} = \int \left(x^{-3}\right)\left(x^{4}\right)dx + C$$

Simplify the integrand using basic laws of exponents ($x^{-3} \cdot x^{4} = x^{1}$):

$$yx^{-3} = \int x dx + C$$

Evaluate the indefinite integral using the fundamental power rule to yield the final general solution:

$$yx^{-3} = \frac{x^{2}}{2} + C$$

4. Homogeneous Differential Equations

A first-order differential equation M(x,y)dx + N(x,y)dy = 0 is classified as homogeneous if both coefficient functions, M and N, possess the exact same mathematical degree. A function is homogeneous of degree n if scaling both independent and dependent variables by a constant factor k allows k^n to be factored out completely, satisfying the property $f(kx, ky) = k^n f(x,y)$. This type is easily recognized because the total sum of the exponents in every single term across the equation is identical.

The Method of Homogeneous Substitution

When an equation is verified as homogeneous, it cannot be separated immediately. Instead, it must be transformed into a separable equation by introducing a change of variables that normalizes the ratio between x and y. Engineers use one of two standard substitutions:

1. **y = vx**, which requires replacing the differential with **dy = v dx + x dv**
2. **x = vy**, which requires replacing the differential with **dx = v dy + y dv**

The substitution y = vx is generally preferred if the dy term has a simpler coefficient, while x = vy is chosen if the dx term is simpler. Once substituted and algebraically expanded, the new equation containing variables x and v is guaranteed to be solvable via standard separation of variables.

Solve the homogeneous equation: $(x^2 + y^2)dx - xydy = 0$

<u>Step 1:</u> Verify homogeneity. Inspect the degrees of each term. In M = $x^2 + y^2$, both terms are of degree 2. In N = -xy, the product of x^1 and y^1 also yields a combined degree of 2. Since all terms are of degree 2, the equation is homogeneous of the second degree.

<u>Step 2:</u> Apply the substitution. Since the dy coefficient (-xy) is a single product term, we will substitute y = vx and dy = vdx + xdv into the equation:

$$\left(x^2 + (vx)^2\right)dx - x(vx)(vdx + xdv) = 0$$

<u>Step 3:</u> Expand and distribute the terms. Multiply out the algebraic expressions carefully:

$$\left(x^2 + v^2x^2\right)dx - vx^2(v\,dx + x\,dv) = 0$$

$$x^2dx + v^2x^2dx - v^2x^2dx - vx^3dv = 0$$

<u>Step 4:</u> Combine like terms. Notice that the $+\ v^2x^2dx$ and $-\ v^2x^2dx$ terms cancel each other out completely:

$$x^2dx - vx^3dv = 0$$

<u>Step 5:</u> Separate variables and integrate. Move the dv term to the right side, then divide by x^3 to group variables:

$$\frac{x^2}{x^3}dx = vdv$$

$$\frac{1}{x}dx = vdv$$

Integrate both sides simultaneously:

$$\int \frac{1}{x}dx + \int vdv$$

$$ln\,x + \frac{v^2}{2} + C$$

<u>Step 6:</u> Substitute back to original variables. The board exam requires the final answer in terms of x and y. Since y = vx, isolate v as $v = \frac{y}{x}$ and substitute it back into your integrated equation:

$$ln\,x = \frac{\left(\frac{y}{x}\right)^2}{2} + C$$

$$ln\,x = \frac{y^2}{2x^2} + C$$

To clear the fraction for presentation, multiply the entire equation by $2x^2$:

$$2x^2 \ln x = y^2 + 2x^2 C$$

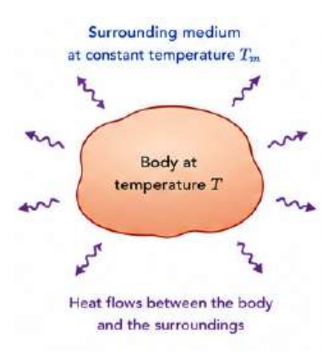
Since 2C is simply another arbitrary constant, we can write it cleanly as C_1, giving the final general solution:

$$2x^2 \ln x = y^2 + C_1 x^2$$

Mechanical Engineering Applications

The board exam heavily emphasizes first-order ODEs applied directly to physical mechanical systems. You must memorize these standard models and understand how to solve them without shortcuts.

1. Newton's Law of Cooling (Thermal Systems)

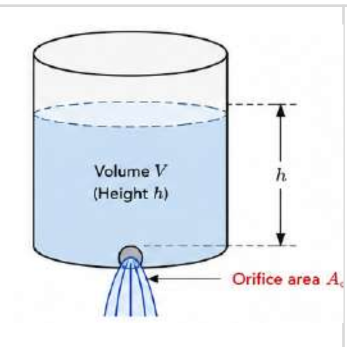


Newton's law of cooling states that the time rate of change of the temperature (T) of a body is directly proportional to the difference between its current temperature and the constant temperature of the surrounding medium (T_m).

The Mathematical Model: $\frac{dT}{dt} = -k(T - T_m)$

Where k is a positive proportionality constant and t represents time.

2. Torricelli's Law for Draining Tanks (Fluid Systems)



Torricelli's Law states that the time rate of change of the volume of water draining out of a tank through a small sharp-edged orifice at the bottom is equal to the volumetric discharge rate of the fluid leaving the system.

The Mathematical Model: $\frac{dV}{dt} = -A_o \cdot v_{exit}$

Where A_o is the cross-sectional area of the opening orifice, and $v_{exit} = \sqrt{2gh}$ represents the theoretical velocity of the exiting fluid stream based on the current height (h) of the fluid surface.

Because the volume V of a uniform tank can be rewritten as the cross-sectional area of the tank (A_t) multiplied by the fluid height (h), the derivative can be rewritten as $dV = A_t dh$. Substituting these geometric variables yields the standard fluid rate differential equation:

$$A_t \frac{dh}{dt} = -A_o \cdot \sqrt{2gh}$$