Combinatorics: Permutations

Permutation is a foundational core topic within combinatorics that describes the distinct ways a specific set of objects can be arranged into a sequential or precise order. Unlike combinations, where selection is the sole criterion, the defining feature of a permutation is that **order matters**. A change in the position of even a single element generates an entirely new arrangement or outcome. In engineering engineering board examinations, permutation problems generally fall into three operational categories: permutations without repetition, permutations with repetition, and cyclic permutations.

Permutation Without Repetition

Permutation without repetition applies when a subset of items is selected from a distinct pool, and each unique item can be utilized at most once within the arrangement. Because elements cannot reappear, the mathematical choices diminish sequentially with each chosen item. The primary formula to evaluate these configurations is defined as: ${}_nP_r = \frac{n!}{(n-r)!}$

Within this governing relationship, n represents the total number of items available to choose from in the initial pool, while r represents the specific number of items to be chosen and arranged. The symbol "!" represents the factorial operation, which multiplies a sequence of descending natural numbers down to one.

Illustration: Find the number of ways to arrange the letters in the word "DOG" into distinct two-letter groups where "DO" is considered a separate arrangement from "OD" and no repeated letters are allowed.

1. Identify the parameters: There are three unique letters available (n = 3), and we are arranging them into groups of two (r = 2).
2. Substitute the values into the formula: ${}_3P_2 = \frac{3!}{(3-2)!} = 6\ ways$
3. Verification: The 6 unique structural arrangements are explicitly mapped as: **DO**, **OG**, **GD**, **DG**, **OD**, and **GO**.

Permutation With Repetition

When elements can be selected multiple times without restriction, the number of options at each position remains constant. Since every choice does not deplete the pool, the calculation builds exponentially. The mathematical relation used to compute permutations with allowed repetition is expressed as: $P = n^r$

In this exponential layout, n serves as the total pool of unique items available, and r designates the total number of positions or slots to be filled in the final sequence.

Illustration: Determine how many 3-letter words can be arranged from the letters in the word "DOG" if repetition of letters is completely allowed.

1. Identify the parameters: The word "DOG" contains three distinct letter options (n = 3), and the desired arrangement size is three letters long (r = 3).
2. Apply the calculation: $P = 3^3 = 27$ ways

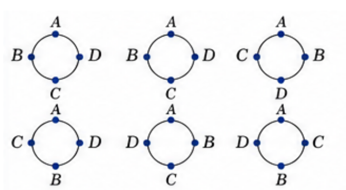
- *D-Group:* DDD, DOO, DGG, DOG, DGO, DOD, DGD, DDO, DDG
- *O-Group:* OOO, ODD, OGG, ODG, OGD, ODO, OGO, OOD, OOG
- *G-Group:* **GGG**, GDD, GOO, GOD, GDO, GDG, GOG, GGO, **GGG**

Cyclic Permutation

Cyclic permutation occurs when objects are arranged in a closed loop or circular formation rather than along a standard linear path. In a linear sequence, positions are absolute (e.g., first, middle, last), meaning a uniform shift of all elements creates a brand new sequence. However, in a circular arrangement, absolute positions vanish; only the relative positions between objects matter. To establish a frame of reference, one object must be fixed in place, reducing the remaining movable objects by exactly one. The mathematical formula for a circular arrangement is written as: $P = (n - 1)!$

Here, n represents the total number of unique objects to be placed around the loop.

Illustration: Calculate how many unique ways 4 people can be seated around a round table.

	Identify the parameter: The number of distinct individuals is four (n = 4). Apply the cyclic calculation: $P = (4 - 1)! = 6\, ways$

Interpretation: Even though there are 4! = 24 linear seating possibilities, rotating the table does not change who sits next to whom. Fixing one person in place yields 6 unique configurations, as shown by the relative sequencing diagrams.

Combinatorics and Probability

A thorough understanding of combinatorics and probability theory is indispensable for engineering board examinations. While permutations deal with ordered arrangements, combinations address groupings where order is irrelevant. Building upon these counting techniques allows engineers to evaluate classical probability and calculate the likelihood of complex independent events through repeated independent trials.

Combinations

A combination is a mathematical selection of a subset of objects chosen from a larger pool where the objects are arranged in an **unordered manner**. The critical conceptual boundary that separates combinations from permutations is the role of sequencing: in combinations, order does not matter. The subset {A, B} is considered identical to the subset {B, A}. The fundamental formula to compute combinations without repetition is modeled as: $_{n}C_{r} = \frac{n!}{(n-r)!\,r!}$

In this algebraic layout, n represents the total number of items available in the initial pool to choose from, and r denotes the number of specific items being selected into the unique group.

Illustration: Find the number of unique combinations of 2 letters that can be made from the word "DOG" without repeating any letters.

1. Identify the parameters: The total number of distinct letter options is three (n = 3), and the desired group size is two (r = 2).
2. Apply the combination formula: $_{3}C_{2} = \frac{3!}{(3-2)!\,2!} = 3\ combinations$
3. Interpretation: Because structural order does not matter, "DO" is mathematically the same as "OD", "OG" is the same as "GO", and "GD" is the same as "DG". Thus, there are only 3 unique subsets.

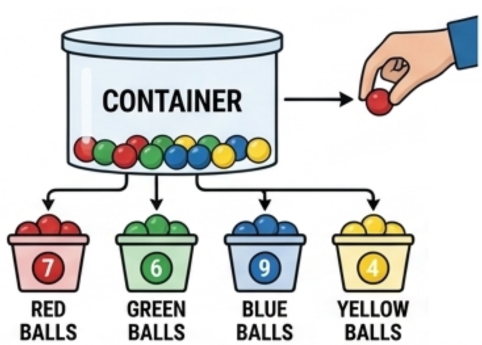
Classical Probability

Classical probability measures the numerical likelihood that a specific random event will occur. It is expressed as a dimensionless ratio between the count of desired successful outcomes and the entire sample space of all possible results. The definitive formula is written as:

$$\text{Probability} = \frac{Number\ of\ Favorable\ ways}{Number\ of\ Possible\ ways}$$

The resulting value can be represented as a fraction, a decimal, or a percentage ranging from 0% (an impossible event) to 100% (a mathematically certain event).

Illustration: A container contains 7 red, 6 green, 9 blue, and 4 yellow balls. If a single ball is chosen at random from the container, what is the probability of choosing a red ball?



1. Compute the total number of possible ways by summing all available elements in the container: $Total\ possible\ ways\ =\ 7\ +\ 6 + 9\ +\ 4\ =\ 26\ balls$
2. Identify the number of favorable ways, which is the total count of red balls available: $Favorable\ ways\ =\ 7$
3. Substitute the values into the primary probability formula: $P_{red} = \frac{7}{26}$
4. Convert the fraction to a percentage: 0.2692 or approximately 27%.

Probability in Repeated Trials (Bernoulli Trials)

A Bernoulli trial is an ideal statistical experiment where an independent action is repeated multiple times, and each individual trial yields exactly two possible mutually exclusive outcomes: a "success" (the desired event occurs) or a "failure" (the desired event does not occur). The probability of success remains perfectly constant from one trial to the next. The Binomial Probability Distribution formula computes the probability (P) that a desired event will happen exactly r times within n total independent trials (such as flipping a coin or guessing on a test repeatedly): $P = {}_{n}C_{r}\, p^{r}\, q^{n-r}$

When utilizing this equation, the variables represent the following parameters:

- n: The total number of independent trials conducted.
- r: The exact number of successful outcomes desired.
- n - r: The resulting number of failed outcomes.
- p: The constant probability that the desired successful event will occur in a single trial.
- q: The constant probability of failure in a single trial, which is always the complement of success (q = 1 - p).

Illustration: If you are taking a 10-question multiple-choice exam where each question has five choices, and you guess randomly on each question, what is the probability of getting exactly 7 correct answers?

1. Establish the basic trial constants: The total number of questions is n = 10, and the target number of correct guesses is r = 7. The number of corresponding failures is n - r = 10 - 7 = 3.
2. Compute the single-trial probabilities: Since there is 1 correct option out of 5 choices per question, the probability of success is p = $\frac{1}{5}$ = 0.20. The probability of picking a wrong choice is q = 1 - 0.20 = 0.80.
3. Substitute these components into the Bernoulli trial formula: $P = {}_{10}C_{7}\, p^{7}\, q^{3}$

Therefore, the mathematical probability of guessing exactly 7 correct answers out of 10 questions is 0.0007864 (or approximately 0.079%).

Advanced Probability Rules and Set Operations

Evaluating compound events and complex structural sample spaces requires a rigorous understanding of basic probability laws and set relationship diagrams. In engineering board examinations, mastering the interaction of multiple events allows professionals to systematically map out safe conditions, operational redundancies, and system breakdown limits.

The Addition Rule of Probability

The Addition Rule is a fundamental probability law used to determine the collective likelihood that at least one of two distinct events will manifest during an experiment. When tracking the probability that event x **or** event y occurs, the calculation must account for whether the two events can take place at the exact same time. The general mathematical relationship is defined as: $P_{(x\ or\ y)} = P_x + P_y - P_{(x\ or\ y)}$

In this structural formulation, the joint probability $P_{(x\ or\ y)}$ is subtracted to prevent double-counting the overlapping region where both events occur simultaneously.

Mutually Exclusive Events

If events x and y are **mutually exclusive**, it is physically or logically impossible for them to occur at the same moment. Consequently, their joint probability is zero ($P_{(x\ or\ y)}$ = 0). Under this condition, the addition rule simplifies to a direct summation of the individual probabilities: $P_{(x\ or\ y)} = P_x + P_y$

Illustration: Consider a college student body consisting of 20% Computer Engineering (CpE) students, 30% Mechanical Engineering (ME) students, and 50% students from other academic majors. If a single student is chosen at random, what is the relative frequency or probability of choosing a student who is either a CpE or an ME major?

1. Identify the event characteristics: A student cannot be simultaneously enrolled as a regular CpE major and an ME major, making these categories mutually exclusive.
2. Apply the simplified addition rule: $P_{(CpE\ or\ ME)} = 0.20 + 0.30 = 0.50 (or\ 50\%)$

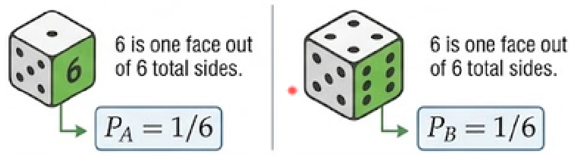
The Multiplication Rule of Probability

The Multiplication Rule is used to evaluate the joint probability that two distinct events, x **and** y, will both occur in sequence or simultaneously. The standard rule changes based on whether the occurrence of the first event alters the likelihood of the second.

Independent Events

If two events are **independent**, the occurrence of one event has absolutely no influence on the probability of the other occurring. Standard everyday examples of independent events include rolling two separate cubical dice at once, drawing a card from a deck and replacing it before drawing a second, spinning a game spinner multiple times, or drawing two marbles from a jar with replacement. For independent scenarios, the joint probability is calculated by multiplying the individual probabilities together: $P_{(x\ or\ y)} = P_x \cdot P_y$

Illustration: If two standard cubical dice, designated as Die A and Die B, are rolled at the exact same time, what is the probability that a value of 6 occurs on both dice?



1. Compute the probability of success for each independent die: Since a cubical die has six sides, the probability of rolling a 6 on Die A is $P_A = \frac{1}{6}$ and on Die B is $P_B = \frac{1}{6}$
2. Apply the independent multiplication rule: $P_{(A\ or\ B)} = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$

Venn Diagrams and Set Operations

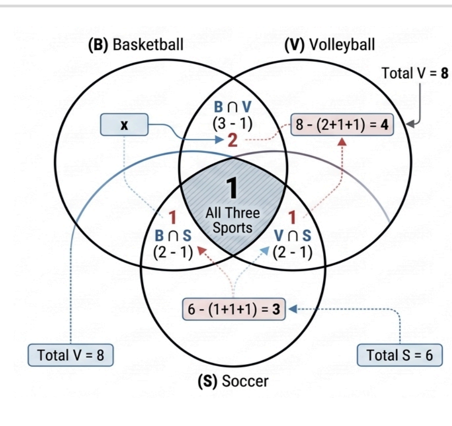
A Venn Diagram is a visual schematic used to represent the relationships, overlaps, and structural differences between various finite sets or event groups. It maps a universal sample space using overlapping circular boundaries, allowing engineers to visually account for every individual element without duplicate counting.

Illustration: A survey was conducted among a group of 15 people regarding their favorite sports to watch. The raw data collected yielded the following breakdown:

- **8 people** watch volleyball (V)
- **6 people** watch soccer (S)
- **3 people** want to watch basketball (B) and volleyball (V)
- **2 people** watch basketball (B) and soccer (S)
- **2 people** watch volleyball (V) and soccer (S)
- **1 person** watches basketball (B), volleyball (V), and soccer (S)

How many people watch basketball only (x)?

1. **Deconstruct the overlapping regions from the inside out:** Always begin with the centermost intersection where all three sets overlap. Here, the number of people who watch all three sports is 1.
2. **Determine the double-intersection regions (subtracting the center):**



Basketball and Volleyball overlap = 3. Subtracting the center (1) leaves 3 - 1 = 2 people who watch only Basketball and Volleyball.

Basketball and Soccer overlap = 2. Subtracting the center (1) leaves 2 - 1 = 1 person who watches only Basketball and Soccer.

Volleyball and Soccer overlap = 2. Subtracting the center (1) leaves 2 - 1 = 1 person who watches only Volleyball and Soccer.

3. **Determine the single-sport outer regions:**
    - For Volleyball (V): The total circle must sum to 8. We know three interior sections contain 2, 1, and 1. The volleyball-only region is 8 - (2 + 1 + 1) = 4.
    - For Soccer (S): The total circle must sum to 6. Knowing the interior sections contain 1, 1, and 1, the soccer-only region is 6 - (1 + 1 + 1) = 3.
    - For Basketball (B): Let the basketball-only region be represented by the unknown variable x.
4. **Set up the global equilibrium equation:** Set the sum of all isolated regions equal to the total survey size of 15:
   15 = x + 2 + 1 + 1 + 3 + 1 + 4
   15 = x + 12
   x = 3

Thus, there are exactly 3 people who watch basketball only.

Engineering Statistics: Variance and Standard Deviation

In engineering design, quality control, and data analysis, measuring the central tendency of a dataset is often insufficient on its own. Engineers must also quantify the variability, dispersion, or spread of the data points around their central average. Understanding structural dispersion helps professionals evaluate manufacturing tolerances, assess experimental uncertainties, and analyze statistical risks on engineering board examinations.

Variance

Variance is a primary statistical metric that measures how spread out a distribution of numbers is, providing a direct mathematical assessment of its overall variability. Conceptually, it represents the average of the squared deviations of each data point from the mean. Squaring the differences ensures that deviations above and below the mean do not cancel each other out, while simultaneously placing a greater statistical weight on extreme outliers.

Depending on the scope of the data collection under evaluation, variance is computed using two slightly different structural equations:

- **Population Variance:** When analyzing a complete, absolute population dataset, the variance (V) is calculated using the following formula: $V = \frac{\Sigma(x-m)^2}{n}$

  Where V represents the population variance, where x denotes each individual score or term, m represents the true population mean, and n represents the total number of terms in the population.

- **Sample Variance:** When an engineer must estimate the variance of a massive population using only a small, representative sample subset, the formula incorporates Bessel's correction to account for potential statistical bias. The sample variance (V_s) is calculated by dividing the sum of squared deviations by n - 1 instead of n:

$$V_S = \frac{\Sigma(x-m)^2}{n-1}$$

Here, m serves specifically as the calculated arithmetic mean of the sample group.

6.2 Standard Deviation

While variance is an excellent mathematical indicator of spread, it is expressed in squared units (e.g., meters2 or seconds^2), which can make direct physical interpretation difficult. To return to the original baseline unit of the measured data, statisticians compute the **Standard Deviation** (SD), which is defined simply as the principal square root of the variance. Because it shares the exact same unit as the underlying data, the standard deviation is the most commonly used engineering metric to measure and report numerical spread. It is expressed mathematically as:

$$SD = \sqrt{Variance}$$

Illustration: Find the standard deviation of the following dataset: 10, 4, 7, 3, and 11.

1. **Count the total number of terms (n):** The dataset contains five unique scores, so n = 5.
2. **Compute the arithmetic mean (m):** Sum all the data values together and divide by the total count: $m = \frac{10+4+7+3+11}{5} = 7$
3. **Calculate the deviations, square them, and find the standard deviation:** Subtract the mean (7) from each individual value, square the results, sum them up, divide by n, and take the final square root:

$$SD = \sqrt{\frac{(10-7)^2 + (4-7)^2 + (7-7)^2 + (3-7)^2 + (11-7)^2}{5}} = \sqrt{10}$$

Evaluating the radical gives an exact standard deviation of $\sqrt{10}$ (or approximately 3.162), quantifying the average baseline dispersion of this specific data group around its mean of 7.