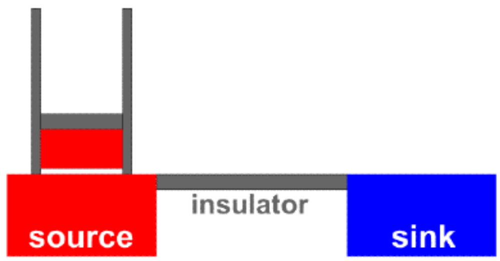
Carnot Cycle

The Carnot cycle is the most efficient cycle. Engines can be compared with this cycle to determine its degree of perfection. The Carnot cycle, proposed by the French engineer Sadi Carnot in 1824, is widely recognized as a reversible cycle and holds significant prominence in thermodynamics. It involves a cyclic process that operates between two heat reservoirs at distinct temperatures.

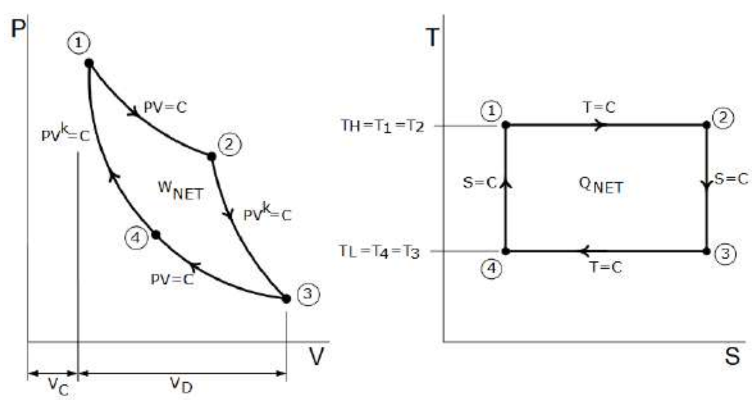
Carnot's Ideal Heat Engine

The Carnot cycle represents an ideal heat engine that exists without any imperfections found in actual engines. However, it cannot be practically implemented in real-world applications. Nevertheless, this theoretical engine serves as a benchmark for evaluating the performance of real engines. It comprises the following components.



- Heat Source: Provides heat at a constant high temperature (T_1) with unlimited capacity.
- Heat Sink: Absorbs heat at a constant low temperature (T_2) without changing its temperature.
- Insulating Stand: Prevents heat transfer to the surroundings.
- Cylinder with Non-conductive Walls and Conductive Bottom: Made with non-conductive walls and a conductive bottom, it holds the working substance and minimizes heat loss. It operates frictionlessly for maximum efficiency.

The working substance is subjected to the following four successive reversible operations so as to complete a reversible cycle. This cycle is called Carnot's Cycle.



Process 1-2 (Isothermal Heat Addition): The cylinder is placed on the source. The piston is allowed to move out infinitely slowly reducing very gradually the weights on the piston. The gas expands extremely slowly. As the gas expands, its temperature tends to fall. But since it is in thermal contact with the heat source therefore it will extract certain heat Q_1 from the source. In this way, the temperature of the gas remains constant at T_1 throughout the
process of expansion.

$$\frac{V_2}{V_1} = \frac{P_1}{P_2} = r_{e(T=C)}$$

$$Q_A = mRT_1 ln\,\frac{P_1}{P_2}$$

Process 2-3 (Isentropic Expansion): The cylinder is placed on the insulated stand and the piston is allowed to move out. The gas expands isentropically from V_2 to V_3, till it falls to T_2.

$$\frac{V_3}{V_2} = \left(\frac{T_2}{T_3}\right)^{\frac{1}{k-1}} = \left(\frac{P_2}{P_3}\right)^{\frac{1}{k}} = r_{e(S=C)}$$

$$W_{2-3} = \frac{mR\left(T_3-T_2\right)}{1-k}$$

Process 3-4 (Isothermal Heat Rejection): The cylinder is placed on the sink and the gas is isothermally compressed until the pressure at the bottom of the volume becomes P_4 and V_4, respectively. The heat Q_2 developed in compression is absorbed by the sink.

$$\frac{V_3}{V_4} = \frac{P_4}{P_3} = r_{K(T=C)}$$

$$Q_R = -\ mRT_3 ln\,\frac{P_4}{P_3}$$

Process 4-1 (Isentropic Compression): The cylinder is placed in the insulating stand and the gas is compressed isentropically till it reaches its initial pressure P_1, volume V_1 and temperature T_1.

$$\frac{V_4}{V_1} = \left(\frac{T_1}{T_4}\right)^{\frac{1}{k-1}} = \left(\frac{P_1}{P_4}\right)^{\frac{1}{k}} = r_{K(S=C)}$$

$$W_{4-1} = -\ \frac{mR\left(T_1-T_4\right)}{1-k}$$

Since: $T_2 = T_1$ and $T_4 = T_3$, hence,

$$\frac{V_3}{V_2} = \frac{V_4}{V_1}$$

$$Isentropic\ expansion\ ratio\ =\ Isentropic\ compression\ ratio$$

$$r_{e(S=C)} = r_{K(S=C)}$$

And,

$$\frac{V_3}{V_4} = \frac{V_2}{V_1}$$

Hence,

$$Isothermal\ expansion\ ratio\ =\ Isothermal\ compression\ ratio$$

$$r_{e(T=C)} = r_{K(T=C)}$$

Thermal Efficiency of Carnot Cycle

$$e_{cc} = \frac{W_{net}}{Q_A} x\ 100\% \ = \frac{|Q_A| - |Q_R|}{Q_A} x\ 100\%$$

Work net:

$$W_{net} = |Q_A| - |Q_R|$$

$$W_{net} = mR(T_1 - T_4) ln\ \frac{P_1}{P_2}$$

Substitute:

$$e_{cc} = \frac{mR(T_1 - T_4) ln \frac{P_1}{P_2}}{mRT_1 ln \frac{P_1}{P_2}} x\ 100\%$$

$$e_{cc} = \frac{T_H - T_L}{T_H} x\ 100\%$$

Or;

$$e_{cc} = 1 - \frac{T_L}{T_H}$$

This equation is one of the most radical generalizations in thermodynamics. The essence of the Second law of thermodynamics and also of the Third law of thermodynamics resides in this result.

Cycle Mean Effective Pressure, P_m

$$P_m = \frac{W_{net}}{V_D}$$

$$P_m = \frac{e_{cc} W_{net}}{(V_3 - V_1)}$$

Observations from Carnot Thermal Efficiency:

$$e_{cc} = 1 - \frac{Q_R}{Q_A}$$

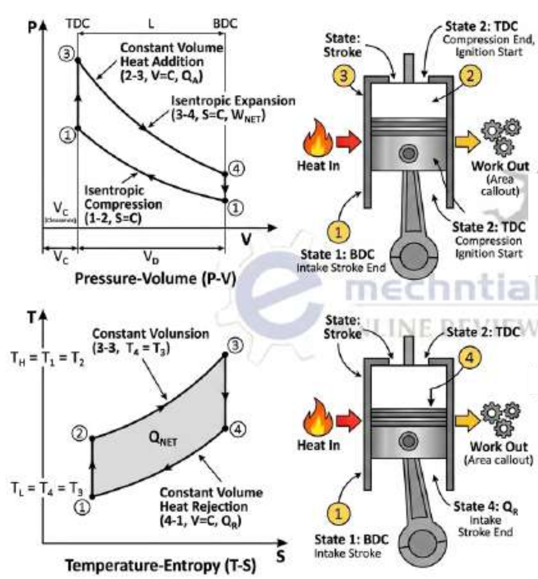
$$e_{cc} = 1 - \frac{T_L}{T_H}$$

1. To increase efficiency, the hot reservoir temperature (T_H) must be higher, or the cold reservoir temperature (T_L) must be lower.
2. The Carnot engine's efficiency depends only on the temperatures of the heat source and sink, not the type of material used.
3. No engine, not even the ideal Carnot engine, can be 100% efficient. Some energy is always lost, and this is a key idea of the **Second Law of thermodynamics.** This is why things like perfect frictionless motion or a perfect vacuum are impossible.
4. Perfect efficiency (e_{cc}=1) would require T_L=0, which is impossible due to the **Third Law of Thermodynamics.**

Otto Cycle

The ideal Otto cycle serves as the theoretical thermodynamic prototype for spark-ignition internal combustion engines, commonly referred to as gasoline engines. Proposed by the German engineer Nicolaus Otto in 1876, this idealized power cycle provides a mechanical baseline for performance, efficiency, and pressure-volume analysis.

Unlike actual operational engines—which feature open chemical systems with mass exchange, varying chemical compositions, and transient thermal losses—the ideal Otto cycle evaluates a closed, fixed mass working fluid under air-standard assumptions to yield clean analytical predictions.



Before analyzing the transient thermal processes of the cycle, the geometry of the reciprocating piston-cylinder assembly must be clearly defined. The physical movement of the piston within the engine cylinder operates between two absolute spatial constraints: **Top Dead Center (TDC)**, which marks the position of minimum cylinder volume, and **Bottom Dead Center (BDC)**, which marks the position of maximum cylinder volume.

- **Clearance Volume (V_C):** The minimum volume enclosed by the cylinder when the piston reaches its highest stroke inflection point (TDC). It is expressed mathematically via the dimensionless clearance ratio (c):

$$c = \frac{V_C}{V_D} = \frac{V_2}{V_D} = \frac{V_3}{V_D}$$

- **Displacement Volume (V_D):** The active physical volume swept or displaced by the piston face as it travels linearly through its stroke length (L) from TDC to BDC:

$$V_D = V_1 - V_2$$

- **Isentropic Compression Ratio (r_k):** The foundational geometric ratio comparing the maximum total internal volume to the minimum internal volume. This parameter fundamentally determines the pressure spikes and thermal ceiling of the cycle:

$$r_k = \frac{V_1}{V_2} = \frac{V_C + V_D}{V_C} = \frac{cV_C + V_D}{cV_C} = \frac{c+1}{c}$$

Process 1-2: Isentropic Compression Process (S = C)

The working fluid (air) starts at BDC with maximum volume V_1 and undergoes mechanical compression to volume V_2 at TDC. Because the process is assumed to be perfectly adiabatic and frictionless, entropy remains constant ($S_1 = S_2$). The absolute fluid temperature and pressure scale drastically according to the classical ideal gas property relationships:

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}} = \left(\frac{V_1}{V_2}\right)^{k-1} = r_k^{\,k-1}$$

Process 2-3: Constant-Volume Heat Addition Process (V = C)

With the piston momentarily stationary at the TDC apex ($V_2 = V_3$), a thermal spike is injected into the system from an external heat source, simulating the instantaneous spark ignition of a gasoline engine. Because no volumetric work occurs $\left(\int PdV = 0\right)$, the entire thermal energy input (Q_A) increases the internal energy of the air:

$$Q_A = mC_V (T_3 - T_2)$$

The absolute pressure scales linearly with the absolute temperature addition:

$$\frac{P_3}{P_2} = \frac{T_3}{T_2}$$

Process 3-4: Isentropic Expansion Process (S = C)

The high-pressure, high-temperature gas forces the piston downward from TDC back to BDC. This constitutes the power stroke of the engine. The expansion is structurally symmetric to the compression stroke, preserving entropy (S_3 = S_4). The expansion ratio $\left(r_e = \frac{V_4}{V_3}\right)$ is geometrically identical to the compression ratio (r_k) because $V_4 = V_1$ and $V_3 = V_2$:

$$\frac{T_3}{T_4} = \left(\frac{P_3}{P_4}\right)^{\frac{k-1}{k}} = \left(\frac{V_4}{V_3}\right)^{k-1} = r_k^{\,k-1}$$

Process 4-1: Constant-Volume Heat Rejection Process (V = C)

The piston reaches BDC ($V_4 = V_1$), and excess thermal energy is rejected out of the system into an external heat sink, simulating the exhaust and blowdown phase. The pressure drops back down to the baseline state 1 conditions. The heat rejected (Q_R) is calculated as:

$$Q_R = m\,C_v(T_4 - T_1)$$

Net Work and Cyclic Integration

The net useful work output (W_{NET}) produced per complete cycle can be evaluated using two different thermodynamic approaches:

1. **The Net Thermal Balance Method:** By applying the First Law of Thermodynamics to a complete cycle ($\Delta U = 0$), the net work is the differences between the absolute magnitudes of heat added and heat rejected:

$$W_{NET} = Q_A - Q_R = mC_v(T_3 - T_2 - T_4 + T_1)$$

2. **The Boundary Work Integration Method** $\left(\oint PdV\right)$**:** By computing the area enclosed within the process curves on the P-V plane, net work equals the expansion work output minus the compression work input:

$$W_{NET} = W_{3-4} - W_{1-2} = \frac{\left(P_2V_2 - P_1V_1\right) - \left(P_4V_4 - P_3V_3\right)}{1-k}$$

Thermal Efficiency Analysis (e_{OC})

Thermal efficiency measures how effectively an engine converts heat input into net work output. For the ideal Otto cycle, this can be progressively derived by substituting the process parameters into the base efficiency statement:

$$e_{OC} = \frac{W_{NET}}{Q_A} = \frac{Q_A - Q_R}{Q_A} = 1 - \frac{Q_R}{Q_A}$$

Substituting the constant-volume heat expressions:

$$e_{OC} = 1 - \frac{mC_v\left(T_4 - T_1\right)}{mC_v\left(T_3 - T_2\right)} = 1 - \frac{T_4 - T_1}{T_3 - T_2}$$

By factoring out T_1 from the numerator and T_2 from the denominator:

$$e_{OC} = 1 - \frac{T_1}{T_2}\left[\frac{\frac{T_4}{T_1} - 1}{\frac{T_3}{T_2} - 1}\right]$$

Since the isentropic process structures enforce that $\frac{T_2}{T_1} = \frac{T_3}{T_4}$ it follows that $\frac{T_4}{T_1} = \frac{T_3}{T_2}$. This equality cancels out the bracketed temperature expression completely, leaving the classic efficiency relationship:

$$e_{OC} = 1 - \frac{T_1}{T_2} = 1 - \frac{1}{r_k^{k-1}}$$

This elegant conclusion proves that the theoretical efficiency of an Otto cycle depends exclusively on the compression ratio (r_k) and the specific heat ratio (k) of the working fluid. Increasing the compression ratio significantly enhances thermodynamic conversion efficiency.

5. Mean Effective Pressure (P_m)

Mean Effective Pressure (P_m) is a valuable parameter used by engineers to assess the work-producing capacity of an engine independent of its physical displacement volume. It is defined as the fictitious, constant gauge pressure that, if acting on the piston face throughout the entire power stroke, would produce the exact same net work output as the actual varying cycle pressure:

$$P_m = \frac{W_{NET}}{V_D} = \frac{W_{NET}}{V_1 - V_2}$$

Using the identity W_{NET} = $e_{OC}Q_A$, this parameter can be stated alternative as:

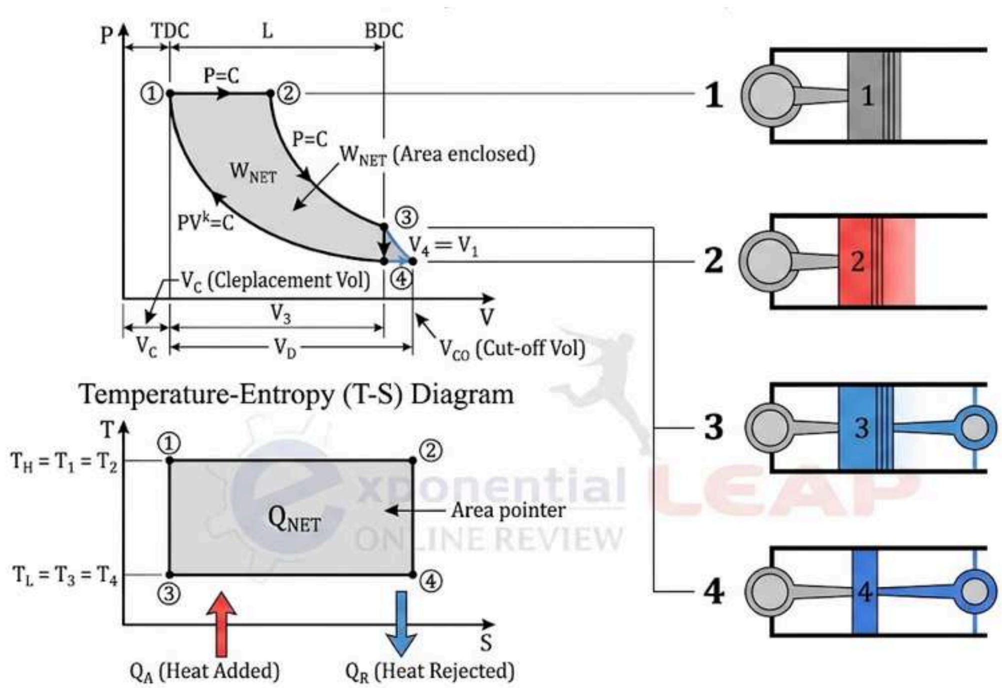
$$P_m = \frac{e_{OC}Q_A}{V_1 - V_2}$$

A high P_m indicate that the engine produces substantial net work relative to its physical size, enabling compact and efficient engine designs.

Diesel Cycle

The ideal Diesel cycle serves as the theoretical thermodynamic prototype for compression-ignition internal combustion engines, commonly referred to as diesel engines. Proposed by the German engineer Rudolf Diesel in 1892, this idealized power cycle provides a mechanical baseline for performance, efficiency, and pressure-volume analysis.

Unlike the Otto cycle, which relies on an external spark plug to initiate rapid combustion at constant volume, the Diesel cycle relies on highly compressed, hot air to automatically ignite liquid fuel as it is steadily injected into the cylinder. Consequently, the heat-addition process occurs over an interval where the piston is actively moving downward, maintaining a nearly uniform pressure boundary.



The physical movement of the piston within the engine cylinder operates between two absolute spatial constraints: **Top Dead Center (TDC)**, which marks the position of minimum cylinder volume ($V_2 = V_C$), and **Bottom Dead Center (BDC)**, which marks the position of maximum cylinder volume (V_1).

- **Clearance Volume (V_C):** The minimum volume enclosed by the cylinder when the piston reaches its highest stroke inflection point (TDC). It is expressed mathematically via the dimensionless clearance ratio (c):

$$c = \frac{V_C}{V_D}$$

- **Displacement Volume (V_D):** The active physical volume swept or displaced by the piston face as it travels linearly through its stroke length (L) from TDC to BDC:

$$V_D = V_1 - V_2$$

- **Isentropic Compression Ratio (r_k):** The geometric ratio comparing the maximum total internal volume at the start of compression to the minimum internal volume at the end of compression:

$$r_k = \frac{V_1}{V_2} = \frac{V_C + V_D}{V_C} = \frac{cV_C + V_D}{cV_C} = \frac{c+1}{c}$$

- **Cut-off Volume (V_{Co}) and Cut-off Ratio (r_C):** The cut-off volume represents the net increase in cylinder space that occurs during the fuel injection phase$\left(V_{Co} = V_3 - V_2\right)$. The dimensionless cut-off ratio (**r_C**) defines the volumetric expansion that takes place during constant-pressure heat addition:

$$r_C = \frac{V_3}{V_2} = \frac{V_2 + V_{Co}}{V_2} = \frac{cV_C + c_o V_D}{cV_C} = \frac{c + c_o}{c}$$

(Where c_o represents the percent cut-off relative to displacement volume, such that $V_{Co} = c_o V_D$.)

- **Isentropic Expansion Ratio (r_e):** In the Diesel cycle, heat addition pushes the piston down past TDC to state point 3 before expansion begins. Consequently, the true expansion stroke takes place over a shorter volumetric interval (V_3 to V_4) than the compression stroke. Because $V_4 = V_1$, the expansion ratio is defined as:

$$r_e = \frac{V_4}{V_3} = \frac{V_1}{V_3} = \frac{V_C + V_D}{V_2 + V_{Co}} = \frac{c+1}{c+c_o}$$

By observing the geometric definitions of r_k and r_C, a highly critical algebraic identity can be established for board exams:

$$r_e = \frac{V_1}{V_3} = \frac{V_1}{V_2} \cdot \frac{V_2}{V_3} = \frac{r_k}{r_C}$$

Process 1-2: Isentropic Compression Process (S=C)

The working fluid (air) starts at BDC with maximum volume V_1 and undergoes mechanical compression to volume V_2 at TDC. Because the process is assumed to be perfectly adiabatic and frictionless, entropy remains constant ($S_1 = S_2$). The absolute fluid temperature and pressure scale drastically according to ideal gas relations:

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}} = \left(\frac{V_1}{V_2}\right)^{k-1} = r_k^{\,k-1}$$

Process 2-3: Constant-Pressure Heat Addition Process (P=C)

Liquid fuel is steadily injected into the highly compressed, superheated air core, burning smoothly as the piston begins its downward stroke. The pressure remains perfectly constant throughout this interval ($P_2 = P_3$). Because this process takes place in an open-boundary moving system, the heat addition (Q_A) must overcome boundary work and is modeled using the constant-pressure specific heat capacity (C_p):

$$Q_A = mC_p(T_3 - T_2)$$

Applying Charles's Law, the absolute temperature scales linearly with the expansion volume:

$$\frac{T_3}{T_2} = \frac{V_3}{V_2} = r_C$$

Process 3-4: Isentropic Expansion Process (S=C)

Fuel injection terminates at the cut-off point (State 3), and the high-pressure gas expands without any heat transfer down to the maximum volume limit at BDC (State 4). Entropy remains constant ($S_3 = S_4$). The expansion ratio $r_e = \frac{V_4}{V_3} = \frac{r_k}{r_C}$ dictates the thermal decay:

$$\frac{T_4}{T_3} = \left(\frac{P_4}{P_3}\right)^{\frac{k-1}{k}} = \left(\frac{V_3}{V_4}\right)^{k-1} = \left(\frac{1}{r_e}\right)^{k-1} = \frac{r_C}{r_k}$$

Process 4-1: Constant-Volume Heat Rejection Process (V=C)

The piston reaches BDC ($V_4 = V_1$), and excess thermal energy is rejected out of the system into an external heat sink during the exhaust blowdown phase. Volume remains constant while pressure falls back to the baseline state 1 conditions. The heat rejected (Q_R) is calculated via the constant-volume specific heat capacity (C_v):

$$Q_R = mC_V(T_4 - T_1)$$

Net Work and Cyclic Integration

The net useful work output (W_{NET}) produced per complete cycle can be evaluated using two separate thermodynamic approaches:

1. **The Net Thermal Balance Method:** By applying the First Law of Thermodynamics to a complete cycle ($\Delta U = 0$), the net work equals the total heat added minus the magnitude of the heat rejected. Note that the specific heat coefficients differ between the processes:

$$W_{NET} = |Q_A| - |Q_R| = mC_p(T_3 - T_2) - mC_v(T_4 - T_1)$$

2. **The Boundary Work Integration Method ($\oint$ PdV):** By computing the area enclosed within the process curves on the P-V plane, net work equals the sum of the expansion work steps minus the compression work step:

$$W_{NET} = W_{1-2} + W_{2-3} + W_{3-4} = \frac{P_2V_2 - P_1V_1}{1-k} + P_2\left(V_3 - V_2\right) + \frac{P_4V_4 - P_3V_3}{1-k}$$

Thermal Efficiency Analysis (e_{DC})

The thermal efficiency measures the percentage of fuel energy successfully converted into mechanical work. For the ideal Diesel cycle, this can be progressively derived by substituting the process parameters into the base efficiency equation:

$$e_{DC} = \frac{W_{NET}}{Q_A} = \frac{Q_A - Q_R}{Q_A} = 1 - \frac{Q_R}{Q_A}$$

Substituting the specific heat relations:

$$e_{DC} = 1 - \frac{mC_v\left(T_4 - T_1\right)}{mC_v\left(T_3 - T_2\right)} = 1 - \frac{1}{k}\left[\frac{T_4 - T_1}{T_3 - T_2}\right]$$

To express this purely in terms of the geometric ratios (r_k and r_c), we define all intermediate state temperatures relative to T1 using our established process rules:

- $T_2 = T_1(r_k)^{k-1}$
- $T_3 = T_2(r_c) = T_1(r_c)(r_k)^{k-1}$
- $T_4 = T_3\left(\frac{r_C}{r_k}\right)^{k-1} = T_1(r_c)(r_k)^{k-1}\left(\frac{r_C}{r_k}\right)^{k-1} = T_1(r_C)^k$

Substituting these back into the temperature fraction yields:

$$e_{DC} = 1 - \frac{1}{k}\left[\frac{T_1\left(r_C^{\,k}\right) - T_1}{T_1(r_C)\left(r_k^{\,k-1}\right) - T_1\left(r_k^{\,k-1}\right)}\right]$$

Factoring out T_1 from the numerator and $T_1\left(r_k^{\,k-1}\right)$ from the denominator results in the classic efficiency expression:

$$e_{DC} = 1 - \frac{1}{r_k^{\,k-1}}\left[\frac{r_C^{\,k} - 1}{k(r_C - 1)}\right]$$

A comparison of this formula to the Otto cycle efficiency shows that the presence of the term $\left[\frac{r_C^{\,k}-1}{k(r_C-1)}\right]$—which is always greater than 1 for any $r_c > 1$—mathematically decreases efficiency. Therefore, **at identical compression ratios, the Otto cycle is more efficient than the Diesel cycle.** However, because diesel engines avoid premature fuel auto-ignition (knocking), they can operate at much higher compression ratios ($r_k \approx 15$ to 20) than gasoline engines ($r_k \approx 8$ to 11), allowing real-world diesel engines to achieve higher overall efficiencies.

Mean Effective Pressure (P_m)

Mean Effective Pressure (Pm) is a valuable parameter used by engineers to assess the work-producing capacity of an engine independent of its physical displacement volume. It is defined as the fictitious, constant gauge pressure that, if acting on the piston face throughout the entire power stroke, would produce the exact same net work output as the actual varying cycle pressure:

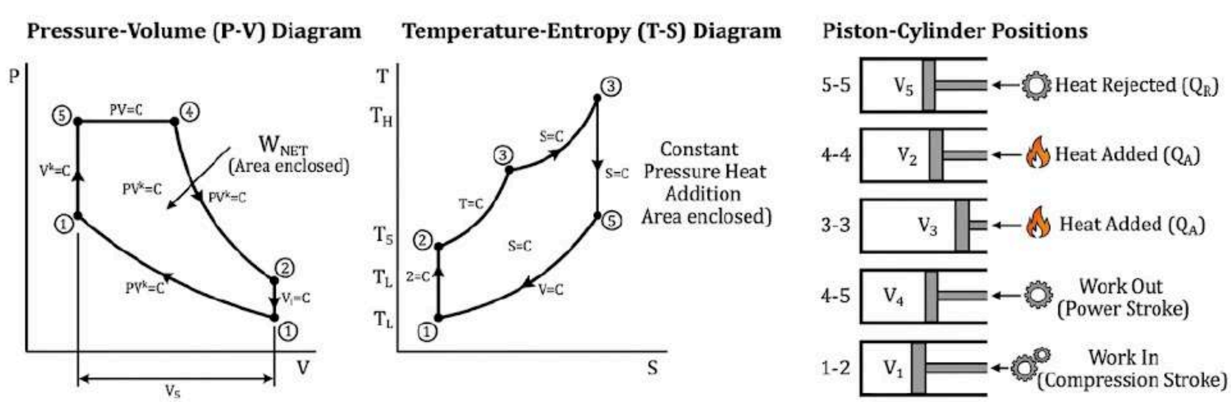
$$P_m = \frac{W_{NET}}{V_D} = \frac{W_{NET}}{V_1 - V_2}$$

Using the identity $W_{NET} = e_{DC}Q_A$, this parameter can be stated alternative as:

$$P_m = \frac{e_{DC}Q_A}{V_1 - V_2}$$

Dual Combustion Cycle

The air-standard Dual Combustion cycle—alternatively designated as the **limited pressure cycle**, **mixed cycle**, **Seiliger cycle**, or **Sabathe cycle**—serves as the ultimate thermodynamic model for modern medium-to-high-speed compression-ignition (diesel) engines.



In a real-world engine, fuel injection is initiated before the piston reaches Top Dead Center (TDC). Because combustion takes a finite amount of time, a portion of the fuel burns almost instantaneously while the piston is momentarily stationary near TDC (modeled as constant-volume heat addition). The remaining fuel burns more progressively as the piston moves downward on its power stroke (modeled as constant-pressure heat addition). By combining characteristics of both the Otto (constant-volume) and Diesel (constant-pressure) cycles, the Dual cycle offers a highly accurate approximation of actual internal combustion engine performance.

- **Compression Ratio (r_k):** The ratio of the maximum volume (at BDC) to the minimum volume (at TDC). Because compression begins at state 1 and ends at state 2:

$$r_k = \frac{V_1}{V_2}$$

- **Pressure Ratio / Explosion Ratio (r_p):** The ratio of pressures across the constant-volume heat addition process (Process 2-3):

$$r_P = \frac{P_3}{P_2}$$

- **Cut-off Ratio (r_c):** The ratio of volumes across the constant-pressure heat addition process (Process 3-4):

$$r_C = \frac{V_4}{V_3}$$

Process 1-2: Isentropic Compression Process (S = C)

The working fluid (air) is compressed reversibly and adiabatically from BDC to TDC. The relationship between states 1 and 2 is governed by standard isentropic equations:

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}} = \left(\frac{V_1}{V_2}\right)^{k-1} = r_k^{\,k-1} \quad \Rightarrow \quad T_2 = T_1 r_k^{\,k-1}$$

Process 2-3: Constant-Volume Heat Addition Process (V = C)

The first phase of combustion occurs instantaneously while the piston is at TDC ($V_2 = V_3$). The pressure and temperature rise rapidly to state 3, governed by the pressure ratio (r_p):

$$\frac{P_3}{P_2} = \frac{T_3}{T_2} = r_p \quad \Rightarrow \quad T_3 = T_2 r_p = T_1 r_k^{\,k-1} r_p$$

Process 3-4: Constant-Pressure Heat Addition Process (P = C)

The second phase of combustion occurs progressively as the piston begins its downward stroke. The pressure remains constant ($P_3 = P_4$) while the volume expands to the cut-off state (V_4), governed by the cut-off ratio (r_C):

$$\frac{T_4}{T_3} = \frac{V_4}{V_3} = r_C \qquad \Rightarrow \qquad T_4 = T_3 r_C = T_1 r_k^{k-1} r_p r_C$$

Process 4-5: Isentropic Expansion Process (S = C)

The high-pressure gas expands adiabatically to BDC ($V_5 = V_1$), performing net work. The expansion ratio $\left(\frac{V_5}{V_4}\right)$ can be rewritten using intermediate state volumes:

$$\frac{V_5}{V_4} = \frac{V_1}{V_4} = \frac{V_1}{V_2} \cdot \frac{V_2}{V_4}$$

Because $V_2 = V_3$, this relationship becomes:

$$\frac{V_5}{V_4} = \frac{V_1}{V_2} \cdot \frac{V_3}{V_4} = \frac{r_k}{r_C}$$

Applying the isentropic temperature relation yields:

$$\frac{T_5}{T_4} = \left(\frac{V_4}{V_5}\right)^{k-1} = \left(\frac{r_C}{r_k}\right)^{k-1} \qquad \Rightarrow \qquad T_5 = T_4\left(\frac{r_C}{r_k}\right)^{k-1}$$

Substituting the expression for T_4:

$$T_5 = \left(T_1 r_k^{k-1} r_p r_C\right) T_4 \left(\frac{r_C^{k-1}}{r_k^{k-1}}\right) = T_1 r_p r_C^{k}$$

Process 5-1: Constant-Volume Heat Rejection Process (V = C)

The cycle is completed by rejecting waste heat to an external sink at BDC ($V_5 = V_1$). The pressure and temperature drop back to state 1:

$$\frac{T_5}{T_1} = \frac{P_5}{P_1}$$

Energy Transfer and Net Work Formulation

Because the heat addition occurs across two distinct thermodynamic phases, the total thermal energy input (Q_A) must be evaluated as the sum of constant-volume and constant-pressure heat transfers:

$$Q_A = Q_{A,V=C} + Q_{A,P=C} = mC_V(T_3 - T_2) + mC_P(T_4 - T_3)$$

The heat rejected (Q_R) during constant-volume cooling is:

$$Q_R = -\, mC_V(T_5 - T_1)$$

$$W_{NET} = mC_V(T_3 - T_2) + mC_P(T_4 - T_3) - mC_V(T_5 - T_1)$$

Thermal Efficiency Derivation (e_{DUAL})

The thermal efficiency of the Dual Combustion cycle is defined as:

$$e_{DUAL} = \frac{W_{NET}}{Q_A} = 1 - \frac{|Q_R|}{Q_A} = 1 - \frac{mC_v(T_5 - T_1)}{mC_v(T_3 - T_2) + mC_p(T_4 - T_3)}$$

Dividing the numerator and the denominator by mC_v, and utilizing the relation for the ratio of specific heats $\left(k = \frac{C_p}{C_v}\right)$, simplifies the expression:

$$e_{DUAL} = 1 - \frac{(T_5 - T_1)}{(T_3 - T_2) + k(T_4 - T_3)}$$

Substituting our derived equations for intermediate temperatures (T_2, T_3, T_4,and T_5) relative to T_1 yields:

$$e_{DUAL} = 1 - \frac{T_1 r_p r_C^{\,k} - T_1}{\left(T_1 r_k^{\,k-1} r_p - T_1 r_k^{\,k-1}\right) + k\left(T_1 r_k^{\,k-1} r_p r_C - T_1 r_k^{\,k-1} r_p\right)}$$

Factoring out T_1 from the numerator and $T_1 r_k^{\,k-1}$ from the denominator results in the definitive, exam-ready efficiency equation:

$$e_{DUAL} = 1 - \frac{1}{r_k^{\,k-1}}\left[\frac{r_p r_C^{\,k} - 1}{(r_p - 1) + k r_p(r_C - 1)}\right]$$

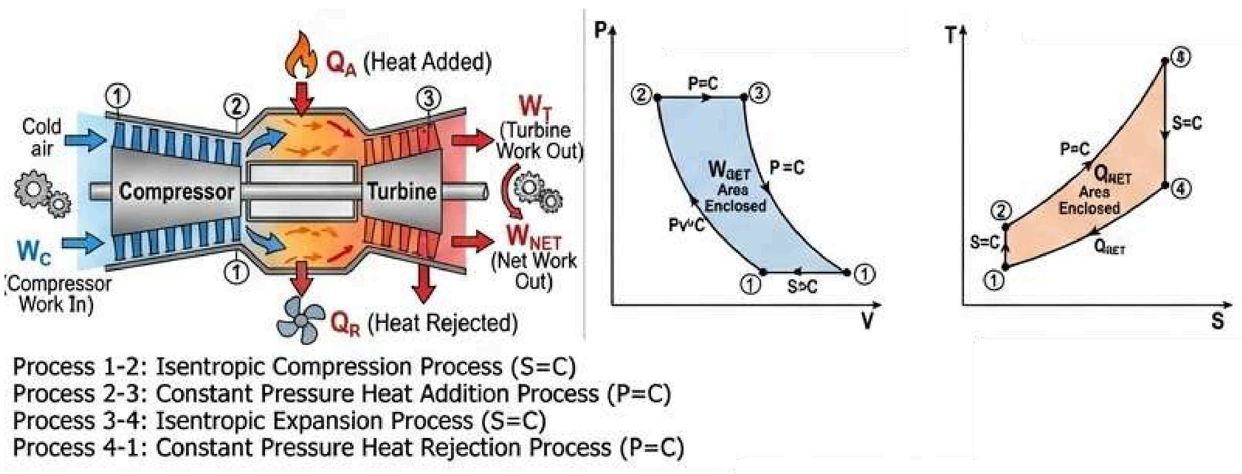
The Dual cycle represents a hybrid thermodynamic model. By adjusting its parameters, it can be simplified mathematically to represent either of its parent cycles:

if $r_p = 1$ (no constant volume combustion)	Diesel Cycle	$e = 1 - \frac{1}{r_k^{\,k-1}}\left[\frac{r_C^{\,k} - 1}{k(r_C - 1)}\right]$
if $r_C = 1$ (no constant pressure combustion)	Otto Cycle	$e = 1 - \frac{1}{r_k^{\,k-1}}$

Brayton Cycle

The ideal **Brayton cycle** (historically referred to as the **Joule cycle** or the **complete-expansion diesel cycle**) serves as the foundational thermodynamic model for gas-turbine power plants and aircraft propulsion systems.

Unlike reciprocating internal combustion engines (like the Otto or Diesel cycles) where compression, combustion, and expansion occur sequentially within the same piston-cylinder volume, the Brayton cycle operates as a **steady-flow (open-cycle)** system. In a physical gas-turbine engine, air is continuously drawn into a rotary compressor, heated in a combustion chamber (or heat exchanger), expanded through a turbine to generate mechanical power, and then discharged. For thermodynamic analysis, we model this as a closed loop using the standard air-cold assumptions.



The ideal air-standard Brayton cycle is composed of four internally reversible, steady-flow processes. The mechanical layout and corresponding state changes are mapped below across the Pressure-Volume (P-V) and Temperature-Entropy (T-S) planes:

- **Process 1-2: Isentropic Compression (S = C)**
  Air in its baseline state is drawn into the rotary compressor and compressed reversibly and adiabatically to a high-pressure state. The entropy remains constant ($S_1 = S_2$), while the temperature and pressure increase significantly.
- **Process 2-3: Constant-Pressure Heat Addition (P = C)**
  The high-pressure air enters a heat exchanger (or combustion chamber) where thermal energy (Q_A or Q_{in}) is transferred at constant pressure ($P_2 = P_3$). This process dramatically raises the temperature and specific volume of the working fluid.
- **Process 3-4: Isentropic Expansion (S = C)**
  The high-temperature, high-pressure gas expands through the turbine to produce work (W_T). The expansion process is reversible and adiabatic ($S_3 = S_4$), resulting in a drop in temperature and pressure.

- **Process 4-1: Constant-Pressure Heat Rejection (P = C)**
  To close the cycle, the exhaust gas passes through a low-pressure heat exchanger (or cooling stage), where waste thermal energy (Q_R or Q_{out}) is rejected at constant pressure ($P_4 = P_1$) back to the heat sink, returning the working fluid to State 1.
- **Compression Ratio (r_k):**

$$r_k = \frac{V_1}{V_2}$$

- **Pressure Ratio (r_p):** This is the primary parameter for gas turbines, representing the ratio of high-to-low pressure boundaries:

$$r_p = \frac{P_2}{P_1} = \frac{P_3}{P_4}$$

Process 1-2 (Compression) Relationships:

$$\frac{T_2}{T_1} = \left(\frac{P_2}{P_1}\right)^{\frac{k-1}{k}} = \left(\frac{V_1}{V_2}\right)^{k-1} \qquad \Rightarrow \qquad \frac{T_2}{T_1} = r_p^{\frac{k-1}{k}} = r_k^{k-1}$$

Process 3-4 (Expansion) Relationships:

$$\frac{T_3}{T_4} = \left(\frac{P_3}{P_4}\right)^{\frac{k-1}{k}} = \left(\frac{V_4}{V_3}\right)^{k-1} \qquad \Rightarrow \qquad \frac{T_3}{T_4} = r_p^{\frac{k-1}{k}} = r_k^{k-1}$$

For an ideal Brayton cycle, because the compression and expansion pressure boundaries are identical, the temperature ratios across both stages are mathematically equal:

$$\frac{T_2}{T_1} = \frac{T_3}{T_4} \qquad \Rightarrow \qquad T_2 T_4 = T_1 T_3$$

Energy, Work, and Back Work Ratio (BWR)

Because the Brayton cycle is analyzed as a steady-flow system, its heat and work interactions are formulated using change in enthalpy (Δh) instead of change in internal energy (Δu). Assuming constant specific heats, these simplify to temperature differences:

- **Heat Added (Q_A):** Occurs across Process 2-3:

$$Q_A = m(h_3 - h_2) = mC_P(T_3 - T_2)$$

- **Heat Rejected (Q_R):** Occurs across Process 4-1.
  To maintain a strictly positive magnitude **$|Q_R|$** for direct use in efficiency equations:

  $$|Q_R| = m(h_4 - h_1) = mC_P(T_4 - T_1)$$

  (Note: If calculated as a signed net heat flow, it can be written as $Q_R = mC_p(T_1 - T_4)$, which yields a negative sign indicating heat leaving the system.)

- **Turbine Work (W_T):** The mechanical power generated by the expanding gas:

  $$W_T = m(h_3 - h_4) = mC_P(T_3 - T_4)$$

- **Compressor Work (W_C):** The power input required to compress the air:

  $$|W_C| = m(h_2 - h_1) = mC_P(T_2 - T_1)$$

The Back Work Ratio (BWR)

In gas-turbine power plants, the compressor is directly coupled to the turbine shaft. This means a significant portion of the work produced by the turbine is immediately consumed to run the compressor. The Back Work Ratio (BWR) represents this parasitic energy consumption:

$$BWR = \frac{W_C}{W_T} = \frac{T_2 - T_1}{T_3 - T_4}$$

Unlike steam power plants where the pump work is minimal (usually under 1% to 2% of turbine output), the BWR for a gas-turbine cycle is exceptionally high, typically ranging from **40% to 80%**. This is because compressing a gas (compressor) requires vastly more work than pumping an incompressible liquid.

Thermal Efficiency Derivation (e_{BC})

The thermal efficiency measures the net work output relative to the heat added to the system:

$$e_{BC} = \frac{W_{NET}}{Q_A} = \frac{W_T - W_C}{Q_A} = \frac{|Q_A| - |Q_R|}{Q_A} = 1 - \frac{|Q_R|}{Q_A}$$

Substituting the steady-flow enthalpy relations:

$$e_{BC} = 1 - \frac{mC_p(T_4 - T_1)}{mC_p(T_3 - T_2)} = 1 - \frac{T_4 - T_1}{T_3 - T_2}$$

To express this in terms of the pressure ratio (r_p), we factor out T_1 from the numerator and T_2 from the denominator:

$$e_{BC} = 1 - \frac{T_1}{T_2}\left[\frac{\left(\frac{T_4}{T_1}\right) - 1}{\left(\frac{T_3}{T_2}\right) - 1}\right]$$

Because we established that $\frac{T_2}{T_1} = \frac{T_3}{T_4}$, rearranging gives $\frac{T_4}{T_1} = \frac{T_3}{T_2}$. Therefore, the bracketed term in the equation simplifies to 1, leaving:

$$e_{BC} = 1 - \frac{T_1}{T_2}$$

By substituting the isentropic relationships, we arrive at the classic efficiency equations for the ideal Brayton cycle:

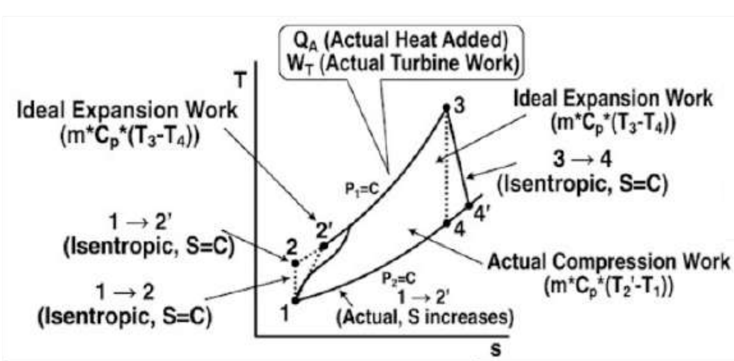
$$e_{BC} = 1 - \frac{1}{r_k^{k-1}} \qquad \text{or} \qquad e_{BC} = 1 - \frac{1}{r_P^{\frac{k-1}{k}}}$$

Real Cycles: Accounting for Irreversibilities

In actual gas turbine systems, components do not operate with 100% thermodynamic perfection. Friction, aerodynamic losses, and non-ideal fluid behavior disrupt the isentropic assumptions.

Component Irreversibilities

- **Fluid Friction & Turbine/Compressor Losses:** Fluid friction occurring within the rotating blades of both the compressor and the turbine causes a drop in efficiency. This friction prevents the compression and expansion stages from being truly adiabatic and reversible. Instead, the working fluid experiences an **increase in entropy** ($S_{2'} > S_1$ and $S_{4'} > S_3$).
- **Pressure Drops in Heat Exchangers:** In a real cycle, viscous resistance causes small pressure drops as the working fluid passes through the heat exchangers ($P_3 < P_2$ and $P_1 < P_4$). However, these pressure drops are very small compared to the mechanical losses within the turbine and compressor, and **they are usually ignored for simplicity** in standard board exam problems.



Deviations on the T-S Diagram

- **Actual Compression (State 1 to 2'):** Because of irreversibilities, the compressor requires more work input to reach the target pressure. The actual discharge temperature ($T_{2'}$) is higher than the ideal isentropic temperature (T_2).
- **Actual Expansion (State 3 to 4'):** Due to internal friction, the turbine extracts less work than the theoretical maximum. The actual exhaust temperature ($T_{4'}$) is higher than the ideal isentropic temperature (T_4).

Isentropic Efficiencies of Turbines and Compressors

To mathematically reconcile the differences between ideal (isentropic) states and actual (real-world) states, we apply **isentropic efficiencies** (η_C and η_t).

Compressor Isentropic Efficiency (η_C)

The compressor isentropic efficiency is defined as the ratio of the work input required in an ideal, isentropic process to the actual work input required to reach the same final pressure:

$$\eta_C = \frac{Isentropic\ Compressor\ Work}{Actual\ Compressor\ Work} * 100\% = \frac{h_2 - h_1}{h_{2'} - h_1} * 100\%$$

Assuming constant specific heats (C_p cancels out):

$$\eta_C = \frac{T_2 - T_1}{T_{2'} - T_1} * 100\%$$

Where:

- T_1 = Compressor inlet temperature (Actual)
- T_2 = Isentropic (ideal) compressor discharge temperature
- $T_{2'}$ = Actual compressor discharge temperature ($T_{2'} > T_2$)

Turbine Isentropic Efficiency (η_t)

The turbine isentropic efficiency is defined as the ratio of the actual work output produced by the turbine to the work output that would be produced in an ideal, isentropic expansion to the same exhaust pressure:

$$\eta_t = \frac{Actual\ Turbine\ Work}{Isentropic\ Turbine\ Work} * 100\% = \frac{h_3 - h_{4'}}{h_3 - h_4} * 100\%$$

Assuming constant specific heats:

$$\eta_t = \frac{T_3 - T_{4'}}{T_3 - T_4} * 100\%$$

- T_3 = Turbine inlet temperature (Actual)
- T_4 = Isentropic (ideal) turbine exhaust temperature
- $T_{4'}$ = Actual turbine exhaust temperature ($T_{4'} > T_4$)

REGENERATIVE POWER CYCLES

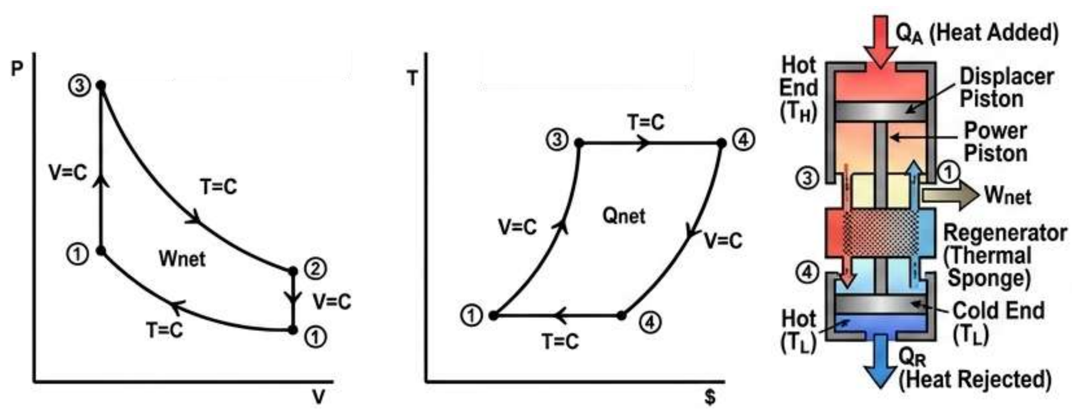
In classical thermodynamics, the Carnot cycle represents the absolute upper limit of thermal efficiency for any engine operating between a m aximum temperature source T_H and a minimum temperature sink T_L. However, the Carnot cycle is highly impractical to construct because its isothermal heat transfer processes must occur extremely slowly, resulting in a very low power output (work per cycle) and requiring high peak pressures.

To bypass these mechanical limitations while still aiming for the maximum possible theoretical efficiency, engineers developed **regenerative gas power cycles**. A **regenerator** is an internal heat exchanger that acts as a thermal storage device. During one part of the cycle, the working fluid passes through the regenerator and transfers its thermal energy to the storage matrix (cooling the fluid). During a later part of the cycle, the fluid passes back through the regenerator in the opposite direction and reclaims that stored energy (heating the fluid).

Both the **Stirling cycle** and the **Ericsson cycle** utilize perfect regeneration to eliminate external heat transfer during their non-isothermal processes. Consequently, they both theoretically achieve the exact same thermal efficiency as the Carnot cycle.

The Stirling Cycle

Historically patented by Robert Stirling in 1816, the Stirling cycle was widely used for hot-air engines before the development of modern internal combustion engines. It is defined as a regenerative thermodynamic cycle consisting of **two isothermal processes** and **two constant-volume (isochoric) processes**.

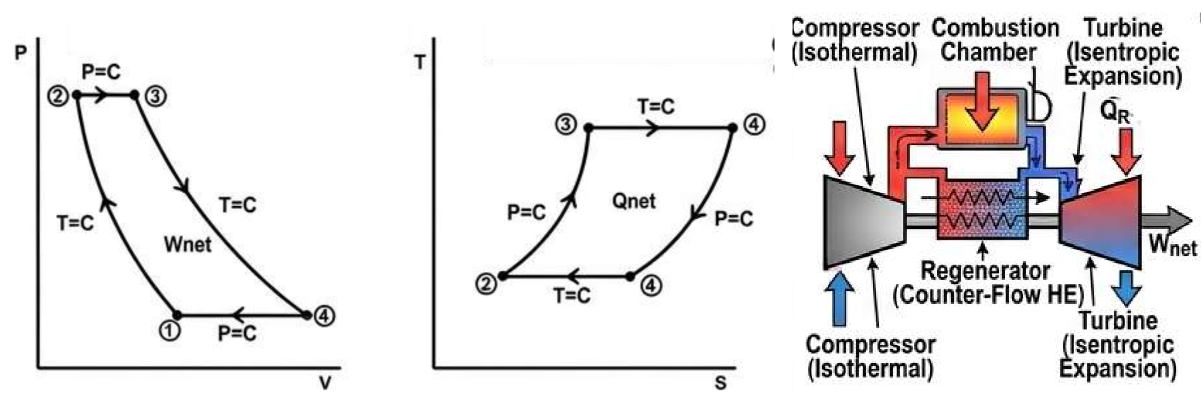


Process-by-Process Analysis

- **Process 1-2: Isothermal Compression (T = C)**
  The working fluid is compressed isothermally at the low temperature limit ($T_1 = T_2 = T_L$) while rejecting heat to an external sink.
- **Process 2-3: Constant-Volume (Isochoric) Heat Addition (V = C)**
  The fluid is forced through the regenerator at constant volume ($V_2 = V_3$). As it passes through, it absorbs the heat stored in the regenerator matrix, raising its temperature from T_L to T_H.
- **Process 3-4: Isothermal Expansion (T = C)**
  The fluid expands isothermally at the high temperature limit ($T_3 = T_4 = T_H$) while absorbing heat from an external source.
- **Process 4-1: Constant-Volume (Isochoric) Heat Rejection (V = C)**
  The fluid passes back through the regenerator at constant volume ($V_4 = V_1$). It transfers its heat to the regenerator matrix, cooling from T_H back to T_L before restarting the cycle.

The Ericsson Cycle

The Ericsson cycle is a regenerative cycle that replaces the constant-volume processes of the Stirling cycle with constant-pressure processes. It consists of **two isothermal processes** and **two constant-pressure (isobaric) processes**. The primary benefit of the Ericsson cycle is that it operates with a smaller pressure ratio for a given volume change, leading to a higher mean effective pressure.



Process-by-Process Analysis

- **Process 1-2: Isothermal Compression (T = C)**
  The gas is compressed isothermally at the cold temperature limit ($T_1 = T_2 = T_L$) while rejecting heat to the environment.
- **Process 2-3: Constant-Pressure (Isobaric) Heat Addition (P = C)**
  The gas flows through the regenerator at constant pressure ($P_2 = P_3$). It absorbs stored heat from the regenerator, raising its temperature from T_L to T_H.
- **Process 3-4: Isothermal Expansion (T = C)**
  The gas expands isothermally at the hot temperature limit ($T_3 = T_4 = T_H$) while absorbing heat from the high-temperature thermal source.

- **Process 4-1: Constant-Pressure (Isobaric) Heat Rejection (P = C)**
  The gas flows back through the regenerator at constant pressure ($P_4 = P_1$). It deposits thermal energy into the regenerator matrix, cooling from T_H to T_L.

4. Thermal Efficiency Equations

By utilizing an ideal regenerator, the heat rejected during the constant-volume (Stirling) or constant-pressure (Ericsson) cooling process is stored internally. This stored heat is identical to the heat required during the heating stage. As a result, no heat is exchanged with the external reservoirs during these transitions.

The only heat added from the external source occurs during the isothermal expansion process (T_H), and the only heat rejected to the external sink occurs during the isothermal compression process (T_L). Because heat addition and rejection occur at constant temperatures, both cycles share the identical thermal efficiency of the Carnot cycle.

Stirling Cycle Thermal Efficiency ($e_{Stirling}$)

$$e_{Stirling} = 1 - \frac{T_L}{T_H}$$

Ericsson Cycle Thermal Efficiency ($e_{Ericsson}$)

$$e_{Ericsson} = 1 - \frac{T_L}{T_H}$$

Where:

- T_L = Absolute temperature of the cold reservoir/sink (in Kelvin or Rankine)
- T_H = Absolute temperature of the hot reservoir/source (in Kelvin or Rankine)